A Simple Circuit for Measuring Complex Impedance

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Introduction

This article shows a method that can be used to measure complex impedance from audio frequencies to very high frequencies (VHF – if tight construction practices are followed and with appropriate diode detectors). All that is required is a known resistance and three AC voltage measurements – for audio frequencies this could be a standard hand-held DVM. The only disadvantage of this method is that only the phase magnitude and not the polarity can be measured. In many cases the phase polarity can be inferred from other known information.

An alternative vector method is also shown that does not require a floating voltage measurement and can resolve phase polarity too. This method is capable of working well into the UHF range with appropriate equipment.

Mathematical Development

The basic circuit is shown in Figure 1. The resistor, R, is a known value and would often be around 50 ohms for RF measurements. For audio frequency measurements it might have to be lowered or raised as needed depending on the magnitude of the impedance being measured. Impedances in the range of about 0.1 to 10 times R can be accurately measured.

The unknown impedance is modeled as a series circuit consisting of an unknown resistance, Rx, and an unknown reactance, jXx. The magnitude of the impedance is Zx.

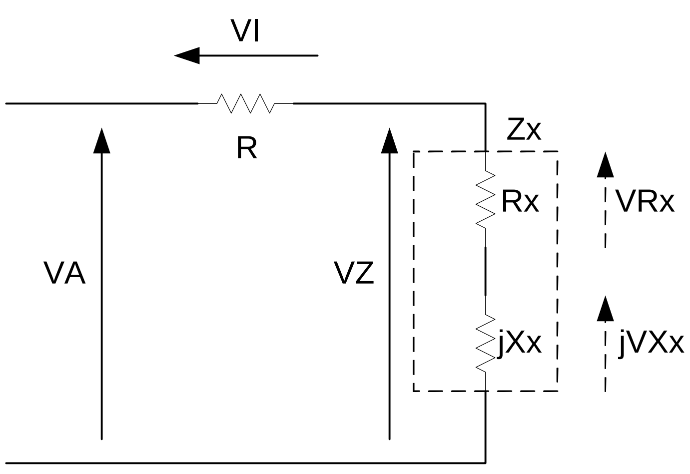

Figure 1: Basic Circuit

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The three voltages that are measured are:

1. VA which is the applied voltage
2. VI which is the voltage across the known resistor and related to the current
3. VZ which is the voltage across the unknown impedance

Although only the magnitudes of these voltages are known, these are actually vectors as shown in Figure 2 which applies to both methods. The angle, θ, is unknown at the moment but can be determined. The vector voltages across the internal resistance and reactance of the impedance are shown for reference but there is no way to directly measure these. The angle, Φ, is used in the alternative vector method discuss later.

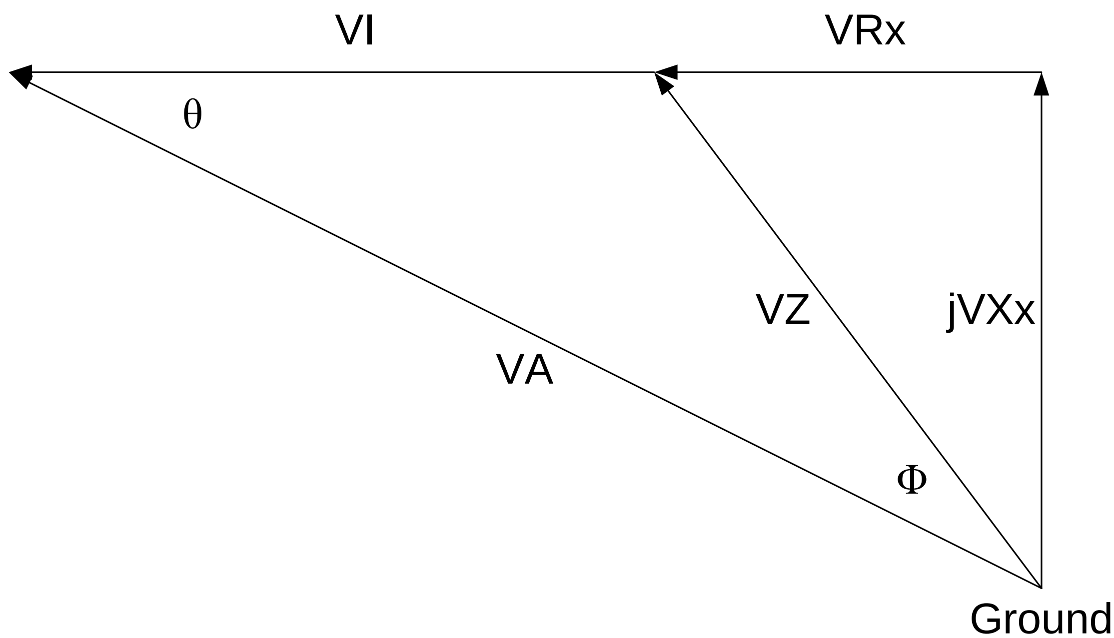

Figure 2: Vector Diagram

The law of cosines is used to calculate the cosine of the angle, θ.

$$\cos(\theta) = \frac{VA^2 + VI^2 - VZ^2}{2 * VA * VI} \qquad \text{Eq. 1}$$

With real measurements there are two things that can go wrong in Equation 1 that need to be artificially corrected. As a reality check the result of Eq. 1 should be between 0 and 1.00 but subtle measurement errors can skew that as follows.

1. Because of small measurement errors it is possible that cos(θ) will be negative – probably only by a small amount. If that happens then use 0.00 as this means that the magnitude of the unknown impedance is purely reactive – in theory this

should never exactly go to zero as the measurement R will cause a small angle shift from 90 degrees. Repeat the measurement using a larger R.

2. It is possible that $\cos(\theta)$ will numerically explode if VI is a very small value – particularly zero. This can happen if the impedance is very large in comparison to R. In such a case the proper thing to do is to substitute 1.00 for the result but the accuracy of the subsequent calculations is going to be very poor. The better solution is to use a larger R so that a definitely measureable voltage across it can be made.

The magnitude of the total impedance (including R) can be calculated as:

$$Za = R * VA / VI \qquad \text{Eq. 2}$$

We note from Figure 1 that the sum of R and Rx can be found by:

$$R + Rx = Za * \cos(\theta) \qquad \text{Eq. 3}$$

Thus, we can solve for Rx by:

$$Rx = Za * \cos(\theta) - R \qquad \text{Eq. 4}$$

Considering possible measurement errors it is conceivable that Rx could compute to be negative which is not likely to be the real situation. The proper thing to do if that happens is to take Rx to be zero. The impedance is purely reactive.

The magnitude of the unknown impedance can be calculated as:

$$Zx = R * VZ / VI \qquad \text{Eq. 5}$$

The magnitude of the unknown reactance can be calculated as:

$$Xx = \mathrm{sqrt}(Zx^2 - Rx^2) \qquad \text{Eq. 6}$$

Considering possible measurement errors it is conceivable that the square root of a negative number might occur. If that happens then Xx should be taken to be zero.

Example

The “unknown” impedance consists of a 30 ohm resistor in series with a 60 ohm reactance which combine to form a 67 ohm complex impedance. The measurement resistor is 50 ohms. The applied voltage, VA, is 1 Vrms. The measured VI is 0.50 Vrms and the measured VZ is 0.67 Vrms. The cosine of theta computes to be 0.800. The impedance computes to be 67 ohms, Rx computes to be 30 ohms, and jXx computes to be j60 ohms. Thus, the method has been shown to work.

Parallel Impedance

The methods used in this article determine the series resistance and reactance. Sometimes the equivalent parallel impedance of a resistance and reactance may be needed. All that is required is a mathematical series to parallel conversion as follows. The concept is to relate the real and imaginary conductance of the parallel network to the conductance of the series network. The numerator and denominator of the series network conductance is multiplied by the complex conjugate of the denominator to put the result in normal form.

$$\frac{1}{Rp} + \frac{1}{jXp} = \frac{1}{Rs + jXs} = \frac{Rs - jXs}{Rs^2 + Xs^2} \qquad \text{Eq. 7}$$

where Rs and Xs are the series values and Rp and Xp are the parallel values.

By equating the real part we have the equivalent parallel resistance and by equating the imaginary part we have the equivalent parallel reactance:

$$Rp = \frac{Rs^2 + Xs^2}{Rs} \qquad \text{Eq. 8}$$

$$Xp = \frac{Rs^2 + Xs^2}{Xs} \qquad \text{Eq. 9}$$

Note that since the phase polarity of Xs was not known then the phase polarity of Xp is also not known but is the same sign. The previous example converts to a 150 ohm resistor in parallel with j75 ohms.

Determining the phase polarity

The sign of the reactance matches that of the angle, θ. This means that an inductive reactance will have positive polarity and capacitive reactance will have negative polarity. If a small capacitive reactance is added in series with the impedance and the magnitude of the reactance increases then the unknown reactance is capacitive. It would be inductive if the magnitude of reactance decreased. A similar concept can be achieved by adding a small inductive reactance with the impedance. The impedance would be inductive if the reactance magnitude increased and capacitive if the reactance magnitude decreased. The additional series reactance should a fraction of the load reactance.

If the unknown impedance is a simple structure of resistance and reactance then the phase polarity can be determined by noting the change in the reactance magnitude as the applied frequency is changed. The reactance is inductive if an increase in frequency causes an increase in the reactance magnitude and capacitive if an increase in frequency causes a decrease in the reactance magnitude. This method can give confusing results if the unknown impedance is a complex network structure involving either numerous reactive components or is a transmission line or antenna or similar structure that has wilding varying reactance with frequency.

Another method involves measuring VI using a time delayed version of VA. The delay should only be just enough for the measurement. This method is similar to the first method of adding a series reactance. If the VI term decreases then the impedance is capacitive.

Vector Method Using Oscilloscope or Vector Voltmeter

Measuring VI requires a floating meter which can have issues, particularly at high frequencies. The setup in Figure 3 can be used to measure VA, VZ, and the angle, Φ, between them – the phase of VZ relative to VA. A vector voltmeter typically provides a direct measure of the phase angle. Using an oscilloscope the peak-peak amplitude of VA and VZ can be measured and the phase angle, Φ, (between the voltages, not the angle of the impedance) can be determined by triggering the scope on VA and measuring the time difference between the zero crossings. The time difference can be related to the phase angle using Equation 10.

$$\Phi = \text{(time\_difference in seconds)} * 360 * \text{frequency in Hz} \qquad \text{Eq. 10}$$

Using the law of cosines and referring to Figure 1 the magnitude of VI can be calculated as:

$$VI = \text{sqrt}(VA^2 + VZ^2 - 2*VA*VZ*\cos(\Phi)) \qquad \text{Eq. 11}$$

Then equations 2 through 6 can be used to determine Rx and Xx. With this approach the polarity of Φ is known. The polarity of Xx is set to be the same as the polarity of Φ noting that the squaring and square root process destroys sign information.

Vector voltmeters often provide the ratio of the channel 2 voltage to the channel 1 voltage which in this case would be (VZ/VA). Then we can calculate

$$(VI/VA) = \text{sqrt}(1 + (VZ/VA)^2 - 2*(VZ/VA)*\cos(\Phi)) \qquad \text{Eq. 12}$$

We then substitute this into a modified version of Equation 1.

$$\cos(\theta) = \frac{1 + (VI/VA)^2 - (VZ/VA)^2}{2 * (VI/VA)} \qquad \text{Eq. 13}$$

If the ratio is given in dB then we first calculate

$(VZ/VA) = 10^{(dB/20)}$ Eq. 14

and then substitute the result of Equation 14 into Equation 12.

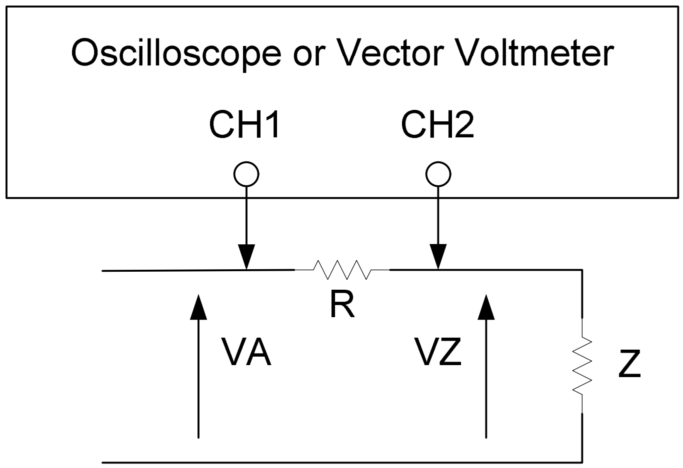

Figure 3: Vector method

An extension to this method that is very applicable at high frequencies uses a vector voltmeter and a power splitter as shown in Figure 4. Note that this is a pure bridge with the only difference in that the scope or meter measures the voltage with respect to ground of each side of the bridge rather than the difficult to measure floating voltage between the arms. The math is identical for this method except that the (VZ/VA) value must be divided by 2 since the math is derived using Figure 3 and VZ is the ratio to one-half of the applied voltage.

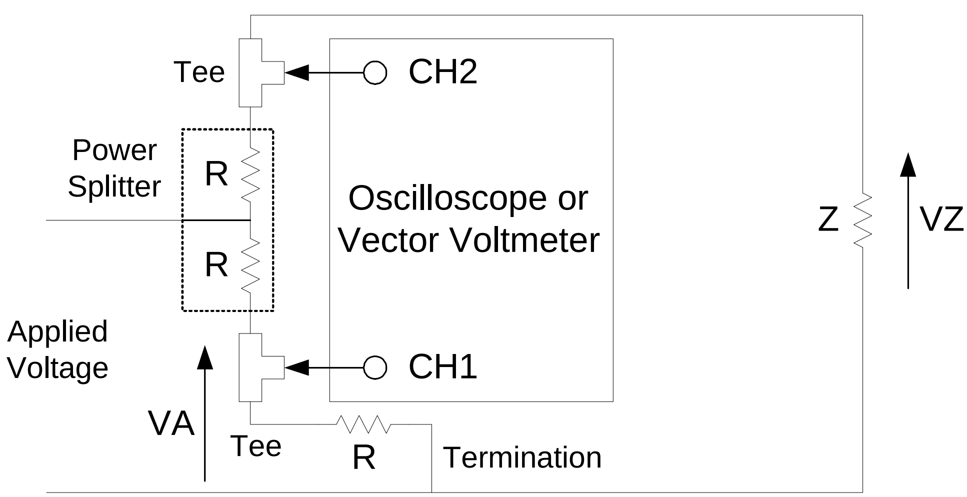

Figure 4: Vector Voltmeter using power splitter

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Some advanced thoughts on the vector math

If the vector voltmeter implementation of Figure 3 is used then the (VZ/VA) is as before. That value should be halved if using the power splitter method shown in Figure 4.

It is convenient to merge Equations 11 and 12 so that only the ratio (VZ/VA) is used.

$$\cos(\theta) = \frac{1 + [1 + (VZ/VA)^2 - 2*(VZ/VA)*\cos(\Phi)] - (VZ/VA)^2}{2*\mathrm{sqrt}[1 + VZ/VA)^2 - 2*(VZ/VA)*\cos(\Phi)]} \qquad \text{Eq. 15}$$

which simplifies to

$$\cos(\theta) = \frac{1 - (VZ/VA)* \cos(\Phi)}{\mathrm{sqrt}[1 + (VZ/VA)^2 - 2*(VZ/VA)*\cos(\Phi)]} \qquad \text{Eq. 16}$$

Equation 5 was

$$Zx = R * VZ / VI \qquad \text{(Eq. 5)}$$

We note that

$$VZ/VI = \frac{(VZ/VA)}{(VI/VA)} = \frac{(VZ/VA)}{\mathrm{sqrt}[1 + (VZ/VA)^2 - 2*(VZ/VA)*\cos(\Phi)]} \qquad \text{Eq. 17}$$

Therefore

$$Zx = R * \frac{(VZ/VA)}{\mathrm{sqrt}[1 + (VZ/VA)^2 - 2*(VZ/VA)*\cos(\Phi)]} \qquad \text{Eq. 18}$$

The total impedance including the series R on the load side of the bridge is ZA.

$$ZA = R / (VI/VA) = \frac{R}{\mathrm{sqrt}[1 + (VZ/VA)^2 - 2*(VZ/VA)*\cos(\Phi)]} \qquad \text{Eq. 19}$$

The real part of the unknown impedance is then

$$Rx = ZA*\cos(\theta) - R \qquad \text{(Eq. 4)}$$

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By combining Equations 4 and 16 we have

$$Rx = ZA*\cos(\theta) = \frac{R * (1 - (VZ/VA) * \cos(\Phi))}{1 + (VZ/VA)^2 - 2*(VZ/VA)*\cos(\Phi)} - R \qquad \text{Eq. 20}$$

which simplifies to

$$Rx = R * \frac{(VZ/VA) - \cos(\Phi)}{2*\cos(\Phi) - (VZ/VA) - [1 / (VZ/VA)]} \qquad \text{Eq. 21}$$

The reactance of the unknown impedance is calculated as before as

$$Xx = \mathrm{sqrt}(Zx^2 - Rx^2) \qquad \text{(Eq. 6)}$$

and forcing the reactance to be zero should the square root of a negative number occur.

The angle of the unknown impedance is found by taking the arctangent of the ratio of the reactive impedance to the resistive impedance.

The following is a fragment of c language code that shows the processing including handling of numerical issues that can arise in measured data.

```
/* c fragment to process vector voltmeter data */
#define SERIES_RESISTOR 1
#define POWER_SPLITTER 2
#define DEG2RAD (3.141592654 / 180.0)
#define SMALL 1E-20

/* Input Variables to c fragment */
double vv_voltage_ratio; /*(VZ/VA)from vector voltmeter */
double vv_angle; /* angle in degrees between VZ and VA */
double rsystem; /* resistance of series resistor or power divider */
int method; /* either SERIES_RESISTOR or POWER_SPLITTER see #defines */

/* Computed outputs from c fragment */
double vv_angle_cosine;     /* cosine of vector voltmeter angle */
double impedance_magnitude; /* in ohms */
double impedance_angle;     /* in degrees */
double impedance_rseries;   /* in ohms */
double impedance_xseries;   /* in ohms */
double temp;                /* temporary variable */

{
if(method = POWER_SPLITTER)
  vv_voltage_ratio /= 2.0;

vv_angle_cosine = cos(vv_angle*DEG2RAD);
```

```
temp = 1.0
     + vv_voltage_ratio * vv_voltage_ratio
     - 2.0 * vv_voltage_ratio * vv_angle_cosine;

if(temp < SMALL)
  temp = SMALL;  /* This handles negative and too small positive */

temp = sqrt(temp);

impedance_magnitude = rsystem * vv_voltage_ratio / temp;

impedance_rseries = rsystem * (vv_voltage_ratio – vv_angle_cosine))
   / (2.0 * vv_angle_cosine) – vv_voltage_ratio
      -(1.0 / vv_voltage_ratio));

temp = impedance_magnitude^2 –impedance_rseries^2;

if(temp < 0.0)
  temp = 0.0;

impedance_xseries = sqrt(temp);

if(vv_angle < 0.0)
  impedance_xseries = -impedance_xseries;

impedance_angle = atan2(impedance_rseries, impedance_xseries)
                / DEG2RAD;
}
```