Crystal Radio Engineering
Mathematical Model of Wire Antenna

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Introduction

Although very simple in structure, the mathematical model of a wire antenna is extremely complicated. There are so many variables that it is basically impossible to accurately model. However, a reasonable model can be derived that provides useful information and understanding. That is essential for doing engineering. An imperfect model is far better than no model. We begin with the simplest possible model (that is always a good starting point!). Then, we refine the model to include more advanced concepts. We keep refining until we have accounted for all significant variables that provides useful information.

A wire antenna consists of some length of wire at some height over the earth. Such a structure will have some capacitance to the earth, some inductance, and some resistance. The resistance term is more complicated than just the simple ohmic resistance of the wire as it includes a series term known as radiation resistance. In very simple terms radiation resistance accounts for a flow of energy between the antenna and free space involving time-dependent electromagnetic fields not considered in a simple RLC circuit. Our initial model is based on frequencies much less than that where the antenna is one-quarter wavelength resonant. For typical crystal radio applications this is not a bad model as it is rare to have the luxury of enough space to construct an antenna approaching one-quarter wavelength in the AM broadcast band.

Capacitance

We start with the capacitance of the antenna to the earth. From the basic laws of physics, a capacitance will exist between any two conductors separated by insulation. Here, we are making the rather crude assumption that the earth is a good conductor. That turns out not to be bad as we will see that the reactance of this capacitance at frequencies in the AM broadcast band is large compared to the actual resistance of the earth.

A brief but excellent derivation of the capacitance between a cylinder (i.e. the antenna wire) and a plane is provided in Reference 1. The summary result is:

$$\text{Capacitance per meter} = \frac{2 \cdot \pi \cdot \varepsilon 0}{\ln(2 \cdot \text{height} / \text{diameter})} \qquad \text{Eq. 1}$$

where height and wire diameter are in meters and ε0 is the permittivity of free space, 8.85 pF/meter. Note that this equation assumes that propagation time over the wire length is negligibly small. This equation also assumes that the earth is a perfect conductor.

Although there is an error when considering the composite antenna, there is no error for an incremental portion that satisfies the assumptions.

A quick example indicates that a 20 meter antenna of #14 wire (1.63 mm dia.) at a height of 3 meters would have a total capacitance of 135 pF at frequencies low enough so that the propagation time over the length is negligibly small. That is not true for broadcast band frequencies and the capacitance will be smaller as will be seen in the plots at the end.

Inductance

On page 141 of Reference 1 the inductance of a wire over a plane is given as

$$\text{inductance per meter} = \frac{\mu 0}{2 * \pi} * \ln(2*\text{height} / \text{diameter}) \qquad \text{Eq. 2}$$

where height and wire diameter are in meters and μ0 is the permeability of free space, 4*π*1E-07 henries/meter. Equation 2 simplifies to

$$\text{inductance per meter} = (0.2\ \text{uH}) * \ln(2*\text{height} / \text{diameter}) \qquad \text{Eq. 3}$$

A quick example indicates that a 20 meter antenna of #14 wire (1.63 mm dia.) at a height of 3 meters would have a total inductance of 33 uH if propagation time is negligibly small. Note that Equation 2 assumes that the current is uniform over the length of the wire and that propagation time is negligibly small. The first part of the assumption can not be true for our antenna as the far end is an open circuit. The equation also assumes that the earth is a perfect conductor. The typical error for our use in antennas for crystal radios falls into the category of not bad. We are not going to worry about it as this is just a starting point and we are going to make refinements. Although there is an error when considering the composite antenna, there is no error for an incremental portion that satisfies the assumptions.

Resistance

The resistance of a wire antenna is the sum of the ohmic resistance including skin effect and the radiation resistance. The resistance of #14 wire including skin effect at a frequency of 1 MHz is around 0.05 ohms per meter. Thus a 20 meter antenna would have a total resistance of 1 ohm. This assumes that the current is uniform over the length which can not really be true since the far end is an open circuit. However, this is not going to be an issue for us as the resistance of the ground will be significantly larger.

The ohmic resistance represents a power loss. Radiation resistance becomes a factor when we consider the coupling of energy between the antenna and electromagnetic fields of free space. The concept is easy to understand in the case of a transmitter which delivers power to the antenna (perhaps significant power in the case of a 50 kW transmitter!) but with efficient design the heating of the antenna is very small – i.e. the power is radiated as an electromagnetic wave instead of being converted to heat – i.e. power is coupled to free space. The transmitter delivers the power to the antenna as some voltage and some in-phase current. Thus, the antenna appears to have a resistance. That resistance is known as the radiation resistance. Unlike ohmic resistance, it does not represent a power loss – it represents a power coupling to free space instead. There is no magic or optimum value for antenna impedance. However, values between single digit ohms and several hundred ohms are the easiest to interface to and so are preferred.

It is not as easy to visualize radiation resistance when the antenna is used for reception. However, the concept of reciprocity applies. Thus, however an antenna appears in the transmitting case, it appears identically in the receiving case. The calculation of radiation resistance is very complicated at the easiest. At this point in our model development we have no way to determine what the radiation resistance of our antenna might be. However, the incremental capacitance and inductance are essential for the next level model.

A finite propagation time model

This model is based on well-developed transmission line theory and treats the wire antenna as a lossy unterminated transmission line where the loss is electromagnetic radiation instead of heat. This model builds on the previous equations for capacitance and inductance and gives us a realistic picture of what goes on. In general, because there are a lot of variables that we are unable to quantify the accuracy of this model is not great. However, the model can be tweaked to provide a good fit to a specific scenario. That is what makes the model useful. It is great at providing us general information for understanding. Understanding what is going on can be more valuable than knowing specific data accurately.

The impedance looking into a transmission line is

$$Z = Z_0 * \frac{(Z_T/Z_0) + \tanh(\alpha + j\theta)}{1 + (Z_T/Z_0) * \tanh(\alpha + j\theta)} \qquad \text{Eq. 4}$$

where:
Z is in ohms and is in general complex
Z_0 is the characteristic impedance of the line – often taken as real but could be complex
Z_T is the termination impedance at the end of the line and could be complex
α is the loss over the length of the line
θ is the phase angle on the line and is equal to 2*π*length/wavelength
j is the square root of -1

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The form of the equation I have chosen to use includes length in the α and θ terms as a matter of convenience. Other forms of the equation have the length as a multiplier on the inside of the tanh term.

If the terminating impedance is an open circuit as it will be for our wire antenna then we can write Equation 4 simply as

$$Z = Z_0 / [\tanh(\alpha + j\theta)] \qquad \text{Eq. 5}$$

For reference:

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \text{Eq. 6}$$

The Euler relations are:

$$e^{j\theta} = \cos(\theta) + j*\sin(\theta) \qquad \text{Eq. 7}$$

$$e^{-j\theta} = \cos(\theta) - j*\sin(\theta) \qquad \text{Eq. 8}$$

These relations let us write Equation 5 as

$$Z = Z_0 * \frac{e^{\alpha+j\theta} + e^{-\alpha-j\theta}}{e^{\alpha+j\theta} - e^{-\alpha-j\theta}}$$

$$= Z_0 * \frac{e^{\alpha} * [\cos(\theta) + j*\sin(\theta)] + e^{-\alpha} * [\cos(\theta) - j*\sin(\theta)]}{e^{\alpha} * [\cos(\theta) + j*\sin(\theta)] - e^{-\alpha} * [\cos(\theta) - j*\sin(\theta)]}$$

$$= Z_0 * \frac{(e^{\alpha} + e^{-\alpha})*\cos(\theta) + j*(e^{\alpha} - e^{-\alpha})*\sin(\theta)}{(e^{\alpha} - e^{-\alpha})*\cos(\theta) - j*(e^{\alpha} + e^{-\alpha})*\sin(\theta)} \qquad \text{Eq. 9}$$

Although Equation 9 looks complicated it really consists of just three components and it is simple to substitute values and reduce it to a simple real plus imaginary result.

For reference:

$$\frac{A + jB}{C + jD} = \frac{A + jB}{C + jD} * \frac{C - jD}{C - jD} = \frac{AC + BD}{C^2 + D^2} + j\frac{BC - AD}{C^2 + D^2} \qquad \text{Eq. 10}$$

Equation 10 gives us a basic method to evaluate the division of complex numbers.

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When the series and shunt losses are low the characteristic impedance of a transmission line is

Z_0 = sqrt(inductance per unit length / capacitance per unit length) Eq. 11

We substitute Equations 1 and 2 into Equation 11 to obtain

$$Z_0 = \text{sqrt}\left[\frac{\dfrac{\mu 0}{2 * \pi} * \ln(2*\text{height} / \text{diameter})}{\dfrac{2*\pi*\varepsilon 0}{\ln(2*\text{height} / \text{diameter})}}\right] \qquad \text{Eq. 12}$$

which simplifies to

Z_0 = 60.0 * ln(2*height / diameter) Eq. 13

Thus, we make use of our previous work on the simple model.

The free-space wavelength in meters for a given frequency in MHz is

$\lambda = 300 / F_{MHz}$ Eq. 14

The propagation velocity on wire antennas is about 95 percent of free space so we use

$\lambda = 285 / F_{MHz}$ Eq. 15

$\theta = 2*\pi*F_{MHz}*\text{length} / 285$ Eq. 16

So far this has not been too complicated. Now, we delve into the challenge of determining the remaining constant, α. This factor, which is a function of frequency, will determine the antenna impedance characteristics. There is no way to calculate this factor. Instead, we will use available information to fit an approximate equation to it.

We begin by noting that the free space impedance of a half-wave dipole antenna is 73 ohms. Our wire antenna is only half of a dipole so the free space impedance will be nominally 37 ohms (rounded half of 73) at a frequency where the wire length is an electrical one-quarter wavelength.

However, our antenna is very close to the ground so the result must be modified. Reference 2 shows a theoretical plot of the impedance of a half-wave dipole over a perfectly conducting plane as a function of height. This plot is reproduced below.

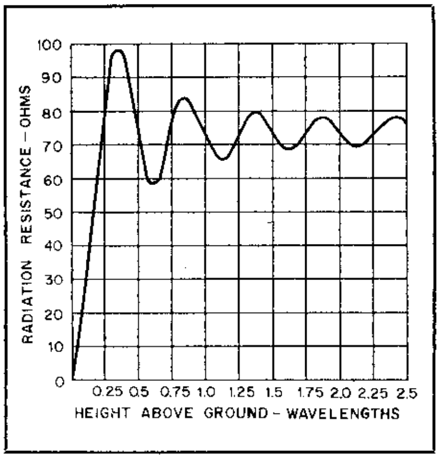

Figure 1: Theoretical impedance of half-wave dipole over perfect ground plane

Reference 2 describes that the impedance of a half-wave dipole over a realistic ground is very little affected by height over about one-half wavelength and the impedance soon attains the 73 ohm free space value.

We will use a simplified exponential to provide us with an approximate impedance of our antenna at the one-quarter wave resonant frequency. We first need a relation that converts the physical height of our antenna to height in wavelengths at the quarter-wave resonant frequency. This is provided by the following relation.

h/λ = height / wavelength = height * F_{MHz} / 300 Eq. 17

$R = 37 * (1 - e^{-k1*h/\lambda})$ Eq. 18

The k1 value is chosen to be 7 because that provides the best least squares fit over the first 0.2 wavelengths of height. We can write this as

$R = 37 * (1 - e^{-0.02333*height*FMHz})$ Eq. 19

Equation 19 provides us with a target to set the appropriate value for α.

At the frequency where the line length is one-quarter wave resonant, the cosine term in Equation 9 goes to zero and the sine term goes to one. This produces a real result with the imaginary part zero. It should be noted for this case that:

$A = 0$

$B = e^{\alpha} - e^{-\alpha}$

$C = 0$

D = 2

Using Equation 10 the above results in a resistance of

$R = Z_0 * (e^{\alpha} - e^{-\alpha}) / 2 = Z_0 * \sinh(\alpha)$ Eq. 20

We invert Equation 20 to obtain the required value for a for a given value of R as follows

$\alpha = \sinh^{-1}(R/Z_0) = \ln[(R/Z_0) + \text{sqrt}[(R/Z_0)^2 + 1)]]$ Eq. 21

Before continuing, let us review where we are at. We know Z_0 from the height and diameter of the antenna wire. We know a target resistance, R, for the antenna impedance at the one-quarter wave resonant frequency based on the height. Equation 21 lets us calculate α for that frequency. We need to expand this so that we can calculate the appropriate α for any frequency.

We make use of the following general knowledge about wire antennas to complete our model. At frequencies well below the one-quarter wave resonance the resistive component of the antenna impedance is practically zero. At frequencies where the antenna is many quarter wavelengths long the resistive component of the antenna impedance converges to Z_0. This suggests that we have a basic scale factor on α and a multiplier based on the number of quarter wavelengths are on the line. This leads us to

$\alpha = k0 * Z_0 * 37 * (1 - e^{-0.02333*height*FMHz}) * (F / Fq)$ Eq. 22

where:
k0 is a constant to achieve the target quarter-wave resonant impedance
Z_0 is the characteristic impedance of the antenna
height is in meters
F is the frequency in MHz
Fq = (71.25 / length) and is the frequency in MHz where the antenna is one-quarter wave resonant

The only unknown in Equation 22 is k0. We calculate k0 when F is equal to Fq. The procedure is as follows:

First we set F_{MHz} = Fq from the above relation

$R = 37 * (1 - e^{-0.02333*height*FMHz})$ (Eq. 19)

$Z_0 = 60.0 * \ln(2*height / diameter)$ (Eq. 13)

$\alpha = \sinh^{-1}(R/Z_0) = \ln[(R/Z_0) + \text{sqrt}[(R/Z_0)^2 + 1)]]$ (Eq. 21)

$k0 = \alpha / (Z_0 * R)$ Eq. 23

Studies done with this equation indicated very good results as long as the frequency was greater than the quarter-wave resonance. For lower frequencies this model gave very poor results – indicating significantly higher impedance than reality and even a negative slope in some cases. A modifier to the model was needed to highly attenuate α for low frequencies only. The factor, $(1 - 1/(1 + 0.14*(F/Fq)^2 + 1.0*(F/Fq)^3 + 0.79*(F/Fq)^4 + 50*(F/Fq)^{24}))$, was determined to significantly improve results. The factors were derived from a least squares fit to an alternative equation for short dipoles provided by reference 3 – shown below modified for only half a dipole.

$$R = 40*\pi^2*(L/\lambda)^2 \qquad \text{Eq. 24}$$

where L is the antenna length and λ is the wavelength, both in meters. This equation is reported to give good results up to about 0.2 wavelengths. A plot showing the fitted model to Equation 24 is shown below. Note that the model has an excellent fit to the theoretical "correct" model for wavelengths shorter than 0.2. The "correct" model under predicts the resistance for longer wavelengths and the effect of the 24th power term comes into play to bring the resistance close to the theoretical 37 ohms at 0.25 wavelengths. This is not exactly what goes on but is a simple to work with fit.

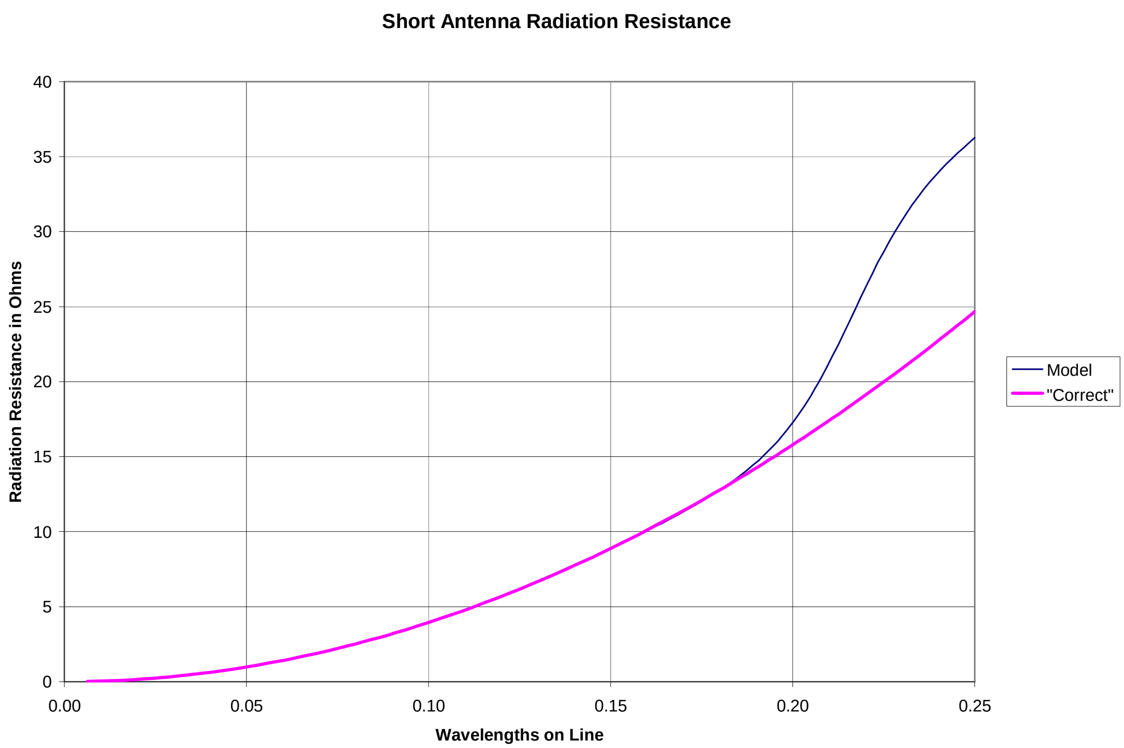

Figure 2: Model fit to short antenna radiation resistance

The complete equation for α is (using n = F/Fq) is shown below.

$$\alpha = k0*Z0*37*(1 - e^{-0.02333*height*F})*n*\left(1 - \frac{1}{1 + 0.14*n^{2} + 1.0*n^{3} + 0.79*n^{4} + 50*n^{24}}\right)$$

Eq. 25

To calculate the approximate impedance of the antenna for any frequency we first compute the α factor for the frequency using Equation 25. Then we compute the positive and negative exponentials using α. Next, we compute theta using Equation 16 and then the cosine and sine terms. We substitute these values into Equation 9 and use Equation 10 to solve. This is best done in a spread sheet for a range of frequencies and the results plotted. A wide range plot showing multiple quarter-wave resonances for a wire antenna is shown in Figure 3.

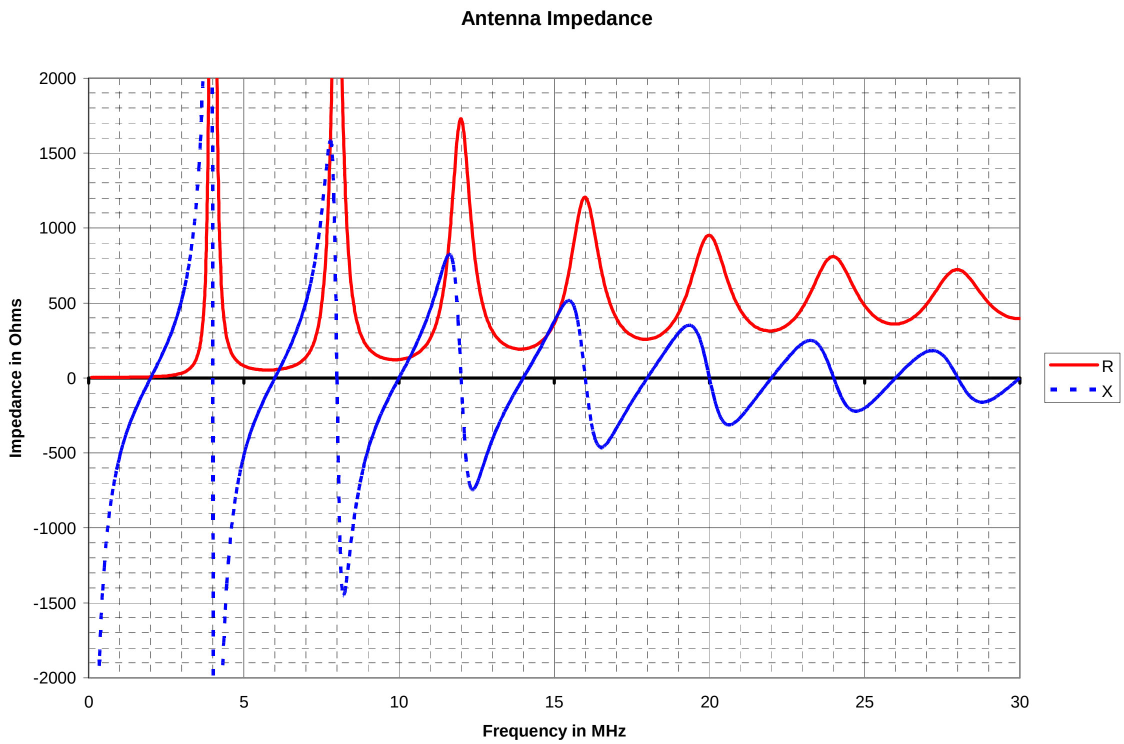

Figure 3: 2 MHz resonate antenna 5 meters off ground

The computed impedance at 2 MHz of this antenna is 7.5 ohms. The impedance at the third harmonic (6 MHz) is 55 ohms – for both resonant frequencies the reactive part is zero. It should be observed that the impedance is at a minimum and purely resistive at odd harmonics of the quarter-wave resonance. The impedance is very high at even harmonics. This figure should be viewed as illustrative rather than accurate. Although the pattern of impedance variations is true, the magnitude could vary considerably depending on actual ground conditions and the proximity of other structures. It is true that the impedance variations become smoother as the number of quarter-waves increases. Only a minimal attempt has been made to correctly model that for this model as the primary interest is in the sub quarter-wave region. More work will be done on the high frequency region at a later date. It is expected that the required effect can be obtained by the appropriate exponent on the n term that immediately follows the exponential in Eq. 25 – but some kind of short power series might be required instead.

Example Antennas

The following plots show the typical impedance for several wire antennas. In each case #14 AWG wire was used and the solid line is the radiation resistance and the dotted line is the reactance. In all cases the radiation resistance is very low. The key feature is the capacitive reactance of the antenna.

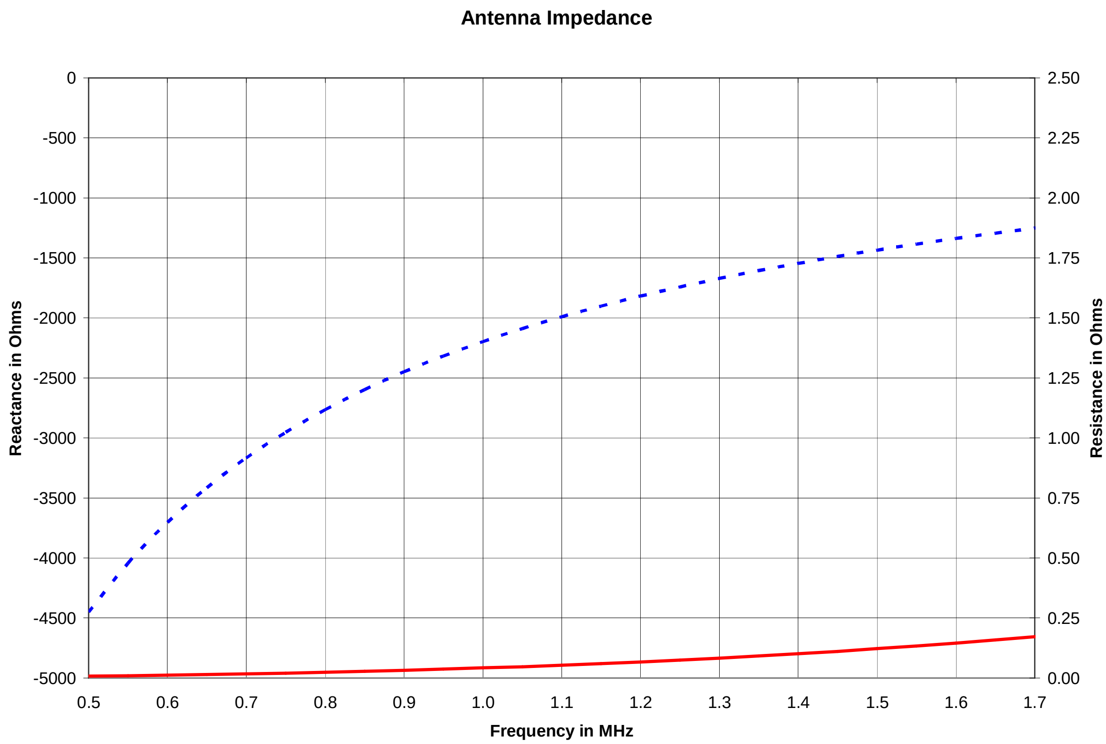

Figure 4: Antenna length = 10 m, height = 3 m

This is about as short as an antenna can be and still be practical. This antenna appears as 73 pF.

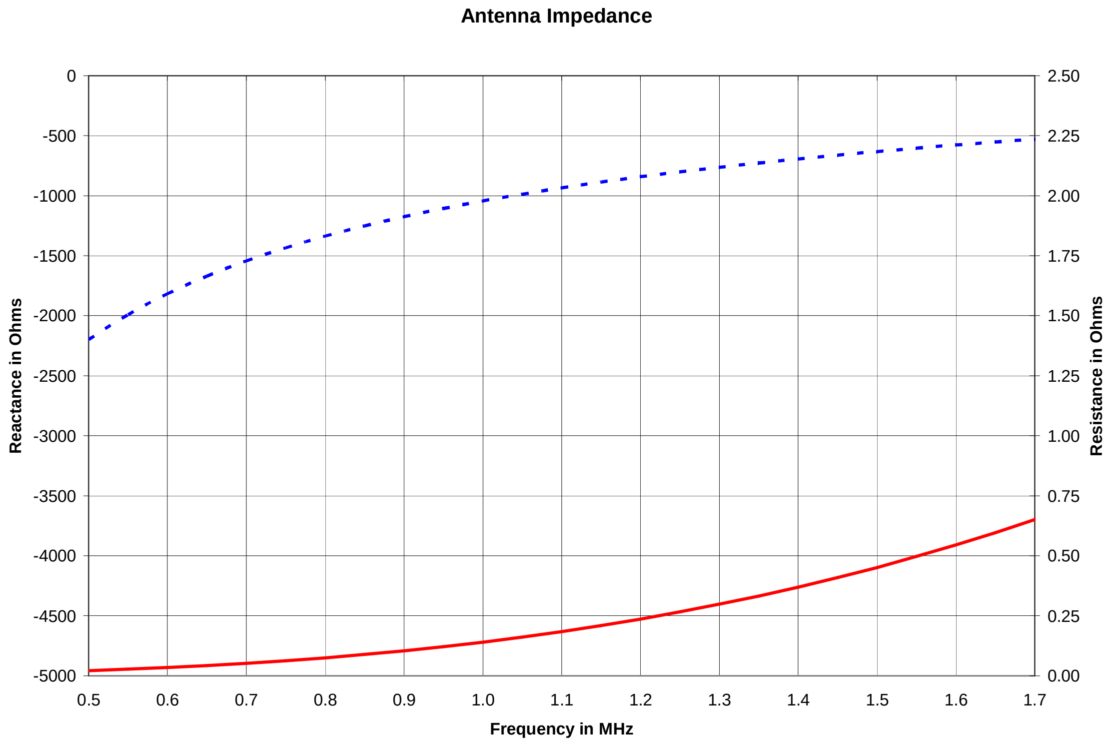

Figure 5: Antenna length = 20 m, height = 3 m

This antenna appears as 145 pF at low frequencies and 184 pF at high frequencies. The change in capacitance with frequency is because the reactance of the series inductance cancels part of the capacitive reactance.

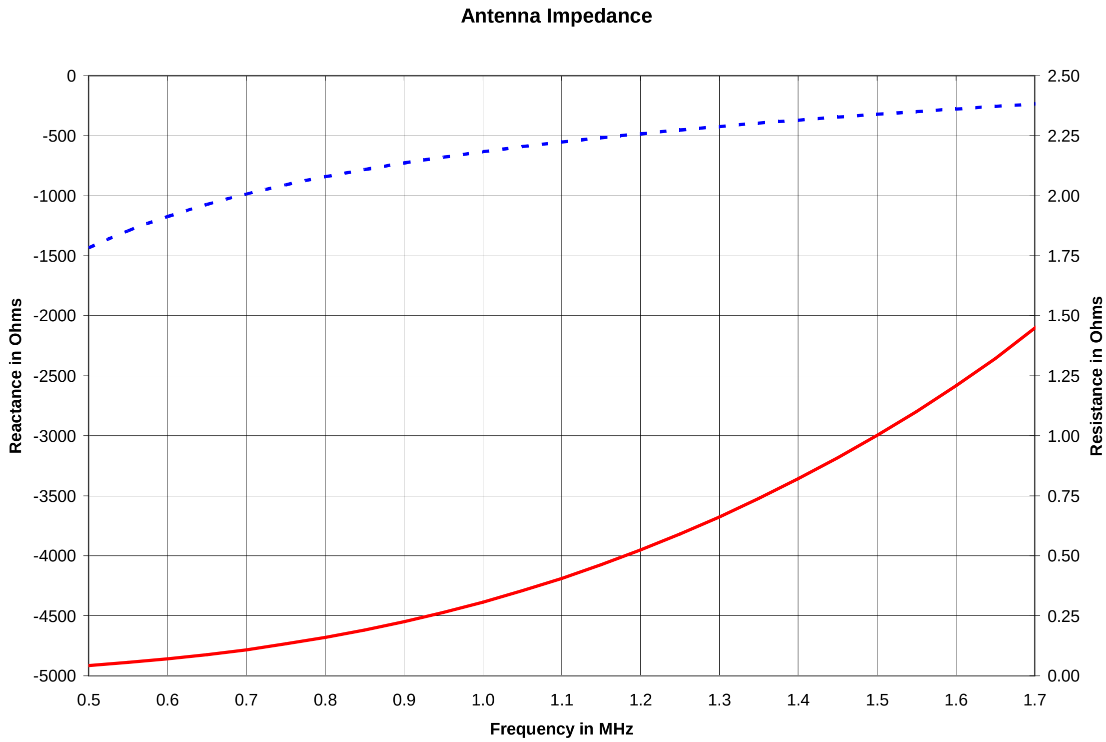

Figure 6: Antenna length = 30 m, height = 3 m

This antenna appears as 220 pF at low frequencies and 375 pF at high frequencies.

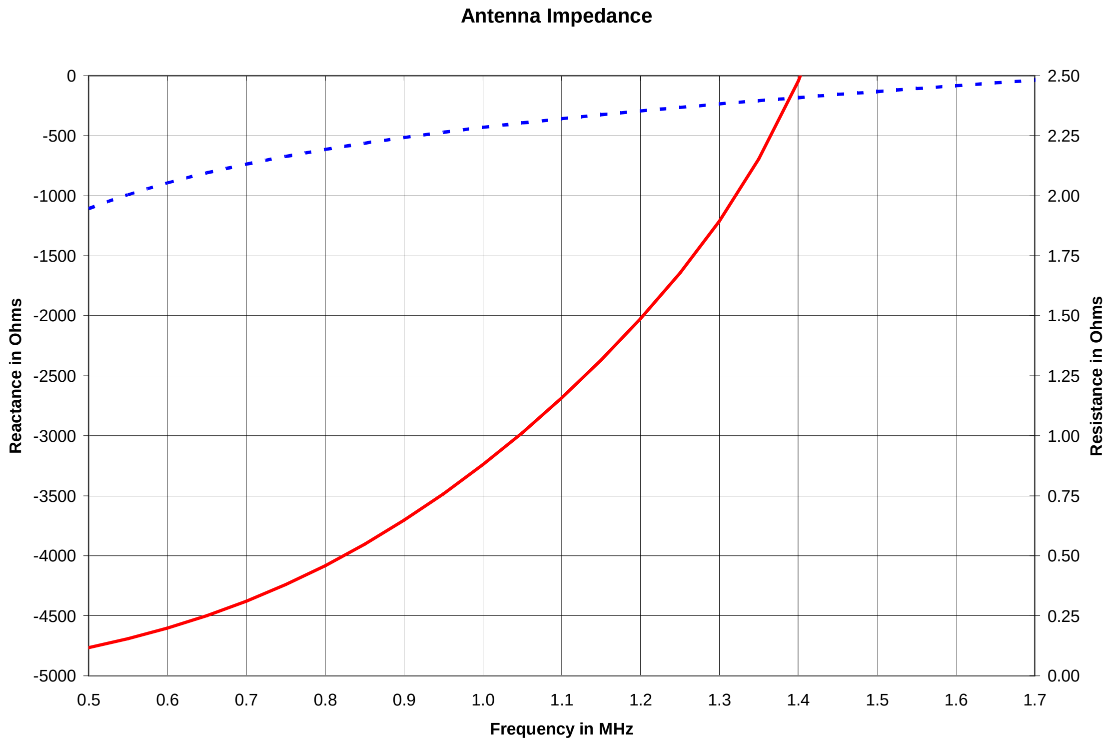

Figure 7: Antenna length = 40 m, height = 5 m

This antenna appears as 290 pF at low frequencies and is almost resonant at 1.7 MHz. This is about as long as an antenna needs to be for excellent reception. Note also that the antenna is higher. The increase in radiation resistance at higher frequencies is an indication that quarter wave resonance is being approached.

References:

1. *Theory and Problems in Electromagnetics*, Schaum's Outline Series, Joseph A. Edminister, McGraw-Hill Book Company, 1979, pages 92-93, 141.

2. *The ARRL Antenna Book*, edited by Gerald L. Hall, The American Radio Relay League, Inc. Newington, CT, 1984, Fourteenth edition, page 2-20.

3. *Antennas and Transmission Lines*, John A. Kuecken, MFJ Enterprises, Inc., 1996, First edition, page 64.