A Mathematical Analysis of the Collapse of the World Trade Center

by Kenneth A. Kuhn
Oct. 31, 2008, rev. Nov. 8, 2008

Introduction

There has been much debate concerning the speed at which the twin towers collapsed on Sept. 11, 2001. Many have felt that the near free fall speed is not possible without some assist such as explosives. Numerous conspiracy theories have resulted. This article will avoid all pros or cons concerning any possible ulterior motive (i.e. conspiracy) in the event and focus only on the physics of a collapsing structure. This article avoids any discussion of why the collapse began as it only deals with the physics that begins afterwards. While this article weakens the explosives theory it changes nothing concerning anything else.

There seems to be remarkably little information on collapsing structures. Maybe that is because so few structures ever fall down unintentionally. That void is probably what leads to the various conspiracy theories. The author has searched the Internet for such information and has found nothing of any use.

There are two basic questions to answer concerning a tall building that collapses unintentionally. If the answers to these questions are in general agreement with the actual WTC collapse then the case for explosives is highly weakened. However, if the answers to these questions are in strong disagreement with the actual WTC collapse then the case for ulterior methods (i.e. explosives) is strengthened.

1. How long would the collapse take?

2. Would the collapse tend to be generally confined to the footprint of the building – i.e. symmetrical collapse?

Discussion of Question 1

For question 1 a basic physics is developed that is easy to understand and to independently verify. I begin with the simplest possible case which makes no claim to accurately represent the WTC collapse. However, this model is the starting point for more advanced models that could provide a more accurate description. I will let interested readers do that as I do not have the time.

For this case we have some tall structure of height, **h**, with a gridded construction that is generally uniform and has total mass, **m**. We begin with total structural failure (caused by whatever means) at the collapse point, **hc**, from the ground. The upper portion of the structure will fall into the lower structure with initial free fall acceleration and we will assume that the impact causes the next lower floor to collapse. This process repeats until the point of collapse is at ground level. The time required for the initial point of collapse to reach the ground will be measured. For this simplistic model we will assume that portions of the structure above the downward propagating collapse point remain within the footprint of the structure and also exist as a simple mass. Everyone should agree that this model is very simplistic. However, it is an easy starting point. The following also contains some simplifying assumptions to make the calculation easy.

We will divide the height of the building into equal distance units, **d**, which could represent the height of one floor or perhaps a fraction of a floor height. The mass of the structure above the collapse point is **m * ((h – hc) / h)** since we are assuming uniformity. The mass of one distance unit is **m * (d / h)**. We will let gravity free fall the upper structure by one distance unit, **d**, and calculate the velocity and momentum. As the next lower unit of **d** collapses it joins the mass of the upper structure and must be accelerated to a velocity that conserves momentum. This preceding process repeats with the new mass free falling one unit of **d** again and again until the propagating collapse reaches the ground. It should be noted that the time we are going to measure is that for the initial collapse point to reach the ground. The upper structure is still falling and will take a little longer to reach the ground.

The calculations are done in metric and **g** represents gravity (nominally 9.8 m/s^2).

Before proceeding we need the solution to the physics problem of how much velocity an object falling at a velocity referred to as, **vo**, gains as it falls by a distance, **d**.

We can write

$$v = g*t + vo \qquad \text{Eq. 1}$$

Noting that distance is the integral of velocity we can write

$$d = \int v*dt = \int (g*t + vo)*dt = g*t^2/2 + vo*t \qquad \text{Eq. 2}$$

Now we are ready to solve the problem: Given: d, g, and vo, Find: t

From Eq. 2 we can write

$$2*d = g*t^2 + 2*vo*t \qquad \text{Eq. 3}$$

Arranging into quadratic form

$$g*t^2 + 2*vo*t - 2*d = 0 \qquad \text{Eq. 4}$$

The solution is

$$t = (-2*vo + \mathrm{sqrt}(4*vo^2 + 8*g*d)) / (2*g) \qquad \text{Eq. 5}$$

which can be simplified to

$t = (-vo + sqrt(vo^2 + 2*g*d)) / g$ Eq. 6

We will accumulate this time for each d unit until the process reaches the ground.

Substituting Eq. 6 into Eq. 1 gives the solution for the new velocity a mass will have after falling distance, **d**, from an initial velocity, **vo**.

$v = sqrt(vo^2 + 2*g*d)$ Eq. 7

Thus, for the initial collapse, the velocity the upper structure will reach after falling the distance of d is

$v = sqrt(0^2 + 2*g*d)$ m/s

It has momentum, **v*m*(h - hc)/h** and it gains mass, **m*(d/h)** – i.e. the mass of the just collapsed floor. To preserve momentum the following equality must hold.

v1 * mass1 = v2 * mass2 Eq. 8

Thus,

v2 = v1*(mass1 / mass2) Eq. 9

For reference,

mass1 = m*((h – hc) / h Eq. 10

and mass2 = m*((h – hc) / h) + m * (d/h)
= m*((h – hc + d) / h) Eq. 11

After preserving momentum the new velocity is:

v * (h – hc) / (h – hc + d) Eq. 12

Note that this new velocity is always retarded from the original velocity. We are now ready to free fall this new mass another unit of d and repeat the process until we reach the ground. Note that the actual mass of the building does not affect the result.

To summarize the steps:

1. Specify the height of the structure, **h**, and the initial point of collapse, **hc**. Start with the initial time, **t**, equal to 0.

2. Use Eq. 7 with **vo=0** to find the velocity after falling one unit of **d**.

3. Use Eq. 12 to retard the velocity to preserve momentum.

4. Use Eq. 6 to calculate the increment of time for the fall and add that to **t**.

5. Repeat steps 2 through 4 (with **vo** equal to the retarded velocity from the previous iteration) until the total drop distance is equal to **hc**.

The results are the total time required and the final velocity.

For accuracy, **d** should be a small part of **h**. It is reasonable to use the height of one floor. Fractional parts of a floor could be used for slightly more accurate calculations but so many approximations have already been made that that seems a waste of time.

An Excel simulation of the collapse based on the above equations was done. The results are shown in Figure 1. The dotted line is a free fall reference for comparison. For the World Trade Center, I used data available on Wikipedia. An average height was taken to be 416 meters and 3.78 meters per floor. For the North Tower I used floor #96 as the collapse point (that is roughly the center of the level that the plane hit – about 363 meters). A spreadsheet was used to do the calculations and the result was that 11.7 seconds was required for the point of collapse to reach the ground as compared to 8.6 seconds for the same point in pure free fall. This time does not include that for the top floor to reach the ground – that would only be a small amount longer. I have searched the Internet for information on the actual collapse time – the total time for the top of the building to reach the ground is roughly 16 seconds give or take compared with a theoretical free-fall time of 9.2 seconds. I am not aware of a time for the collapse point – that would probably be hard to do since that point becomes lost in the debris.

The conclusion of this analysis is that rapid collapse is to be expected. I would expect that a more refined model that accounts for some of the upper mass moving outside of the building footprint to result in a slightly longer collapse time as that would reduce the momentum of the upper mass thus lowering the acceleration of the collapse.

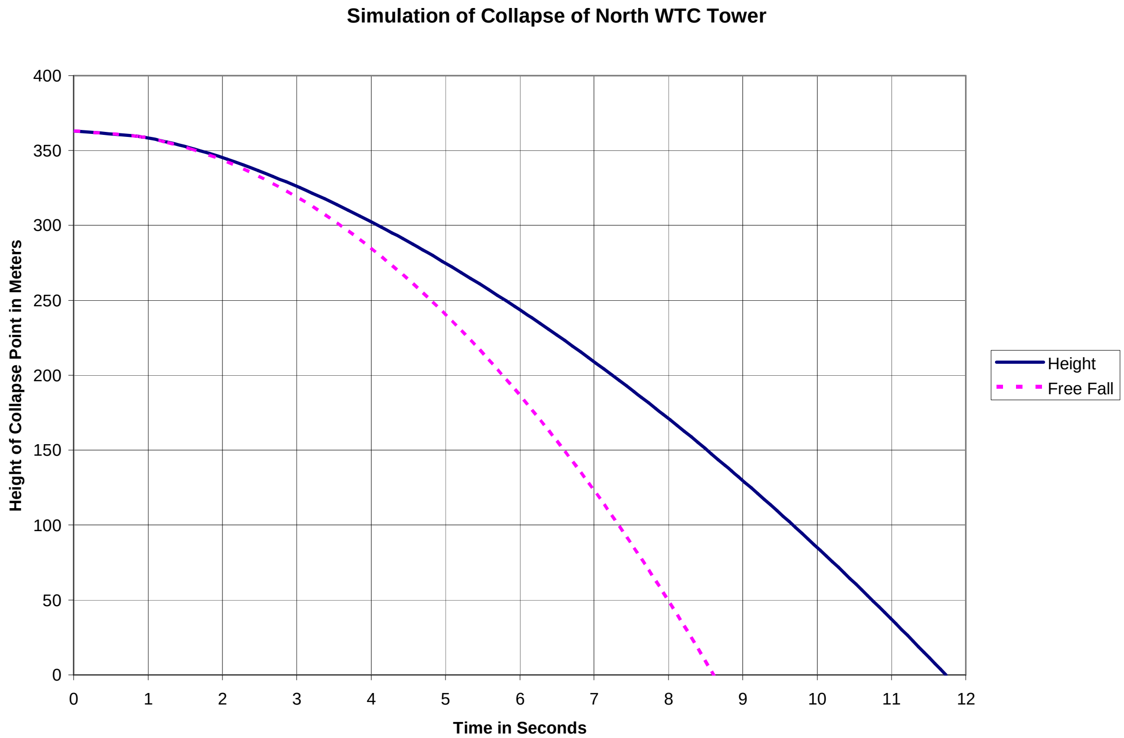

Figure 1: Simulation of Collapse of North WTC Tower

A more complete model

The previous model was the simplest possible. A more complete model would include the effect of portions of the point above the collapse moving outside the building footprint, the energy required to collapse each floor, and the tapered construction that makes lower floors stronger than upper floors. The inclusion of such factors will no doubt increase the total collapse time. I personally do not think it will change a lot but I will let the math point the way. My intention is to add these model extensions and post a revised article by ***January, 2009*** *but I work on many things and time is the issue. I also plan to post the Excel spreadsheets so interested persons can tweak parameters where they disagree from me. I agree with the complaints against NIST who publishes results without showing methods. I will always show methods.*

Discussion of Question 2

I have only a simplistic math for this section and that is that gravity is down. I have been unable to derive any expression for a lateral force that would skew the fall to one side. There really should not be any lateral forces in a building. Any such forces would be the result of deviations from perfect 90 degrees – i.e. related to the cosine of the vertical angle of the structure. Buildings are constructed to high accuracy.

Although a tall building and a tree might seem to have similarities – at least visually, they are entirely different structures. The building is mostly hollow and consists of a gridded frame that contains the forces on the building. If the grid work is interrupted then forces can attain high levels and collapse will occur. Once collapse starts then completion is likely. A gridded structure is capable of collapsing unto itself while a tree is a solid structure that can't and must fall over. A building is not required to fall over although it may. I have seen footage of a twelve story building that did fall over just like a tree because of a foundation collapse on one side. The famous leaning tower of Pisa is a related example although it is not a gridded structure. In both of these cases the failure point was below ground level. The failure point for the World Trade Center was high off the ground.

My belief (and it can only be a belief because I have no hard math – only conjecture) is that the momentum vector from the floors above the collapse point was very strongly in the downward direction. The energy was many times that of what the lower structure could support. Thus, all of the lower structure collapsed simultaneously on a floor by floor basis adding to the downward momentum. There was little opportunity for a non-symmetrical failure that might skew the fall to one side. If the collapse had started very near the top floor (I consider 96 is not near enough) then I believe that some non-symmetry could have occurred that might have resulted in those floors falling out of the footprint of the building and it is conceivable that the collapse might have slowed – perhaps even halted.

I think the answer to this question depends on where the collapse begins and three cases are possible – one for collapse of the foundation, another for a collapse at any floor up to near the top floor, and the last case would be for a collapse that began very near the top floor. For the two WTC towers and including the collapse of WTC7, my conclusion is that those buildings should be expected to fall mostly within their foot prints as they did. It should be noted that all three fell with a relatively small skew from their footprints and the top of one of the towers did fall to one side.

Conclusions

I am convinced that the World Trade Center buildings collapsed following the rules of normal physics and without any assistance of explosions or anything else. As I stated in the beginning, this article weakens the explosives theory but changes nothing else concerning any possible conspiracy.

I am sure that the strong advocates for explosives will find some angle of attack on my analysis or some bizarre concept that I omitted that "could" greatly alter or invalidate my results. I can only speculate but imagine something like how the Reynolds number of concrete dust from the collapse interacts with the yield strength of steel. For such, I have no answer nor would I even attempt an answer. Life is too short to waste. Speculation does not take precedence over physics.