Biquad Filter

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The biquad filter implements both a numerator and denominator quadratic function in *s* thus its name. All filter outputs have identical second order denominator in s and the appropriate numerator terms will exist for each particular filter. A complete biquad as shown in Figure 1 can simultaneously implement a low-pass filter, a high-pass filter, a band-pass filter, and a band-reject or notch filter. The band-reject circuit is usually omitted if that function is not needed. The natural frequency in radians per second is 1/RC and the damping ratio is determined by R_2 and R_3. The circuit has the useful feature that tuning and damping are independent – also known as orthogonal.

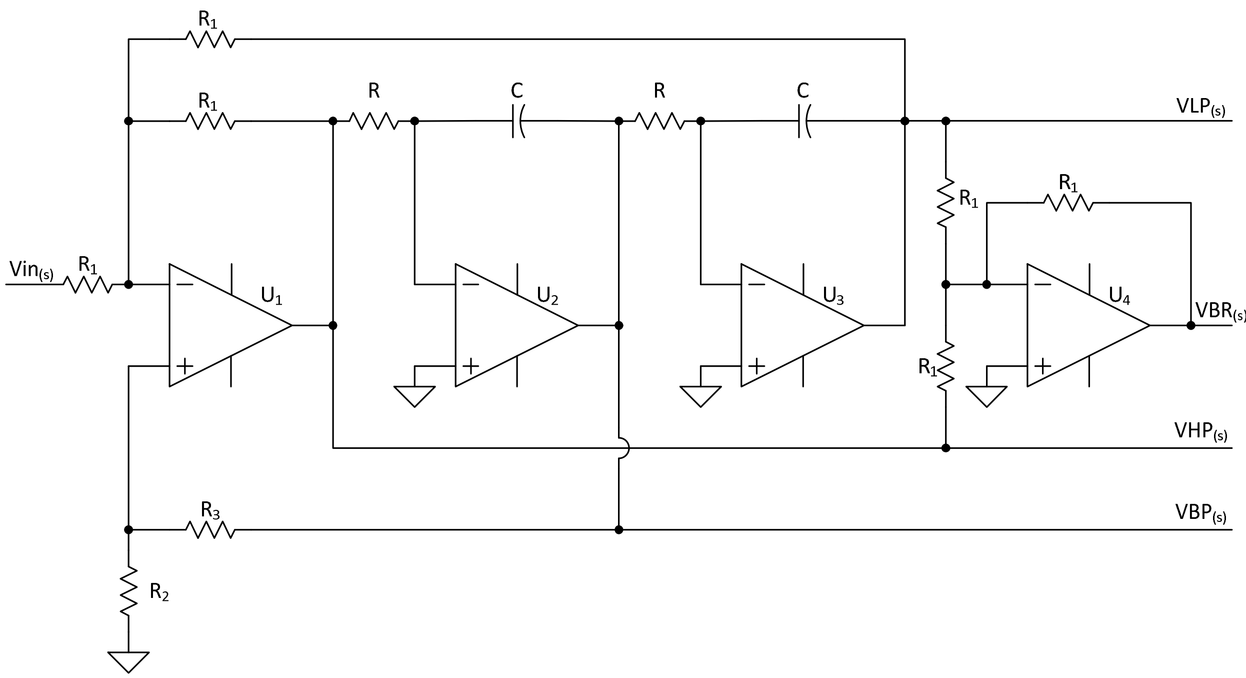

Figure 1: Complete Bi-Quad Filter Schematic
V_{LP} is the low-pass filter output
V_{BR} is the band-reject or notch filter output
V_{HP} is the high-pass filter output
V_{BP} is the band-pass filter output

Analysis

The circuit might at first appear complicated but can be into functional partitions each of which is very simple. All instances of R_1 are the same ohmic value – typically 10K but other values could be used. The two resistors labeled R and the two capacitors labeled C are used in tuning the natural frequency and are always the identical value. A voltage divider is constructed using R_2 and R_3 to apply a selected amount of positive feedback to adjust the damping. U_1 is an inverting summer for two of the inputs and a non-inverting amplifier for the third input. U_2 and U_3 are inverting integrators. U_4 is an inverting summer. The output of U_1 is $V_{HP(s)}$. The output of U_2 is $V_{BP(s)}$. The output of U_3 is $V_{LP(s)}$. The output of U_4 is $V_{BR(s)}$. The (s) subscript will be omitted from the following for convenience.

The output of U_1 is the inverting sum of V_{in} plus V_{LP}. Assuming that V_{in} comes from a zero impedance source then the non-inverting gain of U_1 is 3 because there is a single R_1 in the feedback and a pair of R_1s both to zero impedance sources so the net resistance of those is half of R_1. The output of U_1 is given by Equation 1.

$$V_{HP} = -V_{IN} - V_{LP} + 3\left(\frac{R_2}{R_2 + R_3}\right)V_{BP} \tag{1}$$

We note that

$$V_{BP} = -\frac{V_{HP}}{RCs} \tag{2}$$

and

$$V_{LP} = -\frac{V_{BP}}{RCs} = \frac{V_{HP}}{R^2C^2s^2} \tag{3}$$

We substitute Equations 2 and 3 into Equation 1 and solve for V_{HP}.

$$V_{HP} = -V_{IN} - \frac{V_{HP}}{R^2C^2s^2} - 3\left(\frac{R_2}{R_2 + R_3}\right)\frac{V_{HP}}{RCs} \tag{4}$$

The transfer function to V_{HP} can now be written after a little rearranging.

$$V_{HP} = \frac{-V_{IN}s^2}{s^2 + \frac{3}{RC}\left(\frac{R_2}{R_2 + R_3}\right)s + \frac{1}{R^2C^2}} \tag{5}$$

The standard form of a second order denominator is given by Equation 6 where ω_n is the natural frequency in radians per second and ζ is the damping ratio.

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{6}$$

Equating the terms of Equation 5 with Equation 6 gives us the analytical solution.

$$\omega_n = \frac{1}{RC} \tag{7}$$

We observe that

$$2\zeta\omega_n = \frac{3}{RC}\left(\frac{R_2}{R_2 + R_3}\right) \tag{8}$$

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We can now solve for ζ.

$$\zeta = \frac{\frac{3}{RC}\left(\frac{R_2}{R_2 + R_3}\right)}{2\omega_n} = \frac{\frac{3}{RC}\left(\frac{R_2}{R_2 + R_3}\right)}{2/RC} = \frac{3}{2}\left(\frac{R_2}{R_2 + R_3}\right) \tag{9}$$

Finally,

$$\zeta = \frac{3}{2}\left(\frac{1}{1 + (R_3/R_2)}\right) \tag{10}$$

With the solution to V_{HP} the solutions to the other outputs are simple. We observe from the circuit that V_{BP} is the integral of V_{HP}.

$$V_{BP} = V_{HP}\left(\frac{-1}{RCs}\right) = \left(\frac{-V_{IN}s^2}{s^2 + \frac{3}{RC}\left(\frac{R_2}{R_2 + R_3}\right)s + \frac{1}{R^2C^2}}\right)\left(\frac{-1}{RCs}\right) \tag{11}$$

We now have the result

$$V_{BP} = \frac{V_{IN}s/RC}{s^2 + \frac{3}{RC}\left(\frac{R_2}{R_2 + R_3}\right)s + \frac{1}{R^2C^2}} \tag{12}$$

We next observe from the circuit that V_{LP} is the integral of V_{BP}.

$$V_{LP} = V_{BP}\left(\frac{-1}{RCs}\right) = \left(\frac{V_{IN}\,s/RC}{s^2 + \frac{3}{RC}\left(\frac{R_2}{R_2 + R_3}\right)s + \frac{1}{R^2C^2}}\right)\left(\frac{-1}{RCs}\right) \tag{13}$$

We now have the result

$$V_{LP} = \frac{-V_{IN}/R^2C^2}{s^2 + \frac{3}{RC}\left(\frac{R_2}{R_2 + R_3}\right)s + \frac{1}{R^2C^2}} \tag{14}$$

The band-reject filter is formed by adding the high-pass response to the low-pass response as shown in Equation 15.

$$V_{BR} = -V_{HP} - V_{LP} = \frac{-V_{IN}[(-s^2) - (-1/R^2C^2)]}{s^2 + \frac{3}{RC}\left(\frac{R_2}{R_2 + R_3}\right)s + \frac{1}{R^2C^2}} \tag{15}$$

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That reduces to the final result

$$V_{BR} = \frac{V_{IN}\left(s^2 + {}^{1}\!/_{R^2C^2}\right)}{s^2 + \frac{3}{RC}\left(\frac{R_2}{R_2 + R_3}\right)s + \frac{1}{R^2C^2}} \tag{16}$$

It is important to know the magnitude of the transfer function for each of the filters when *s* is equal to the natural frequency as shown below. Note that in a band-pass filter the Q may be significant (say around 10) so be aware that the gain of the circuit at the center frequency will be Q.

Filter type	**Response at natural frequency**
Low-pass	Q
High-pass	Q
Band-pass	Q
Band-reject	0

Design

We are typically given the filter specification in terms of the natural frequency, F_n, in Hertz and Q rather than in radians/second and ζ.

The first step is to determine a good value for the tuning capacitance, C based on the design note, *Choosing Resistors and Capacitors for Op-Amp Active Filters* written by this author and available on the EE431 class website. The following equation comes from that note and provides us with the geometric mean of good values of capacitance in farads to use given the natural frequency, F_n, in Hertz. Reasonable capacitance values extend from less than one-third to over three times this value so we always round the calculation result upwards or downwards to a convenient standard value of capacitance.

$$C = \frac{4 \times 10^{-7}}{\sqrt{F_n}} \tag{17}$$

With the capacitance determined the next step is to calculate the required resistance, R, to achieve the desired F_n.

$$R = \frac{1}{2\pi F_n C} \tag{18}$$

Q and ζ are related as shown in Equation 19.

$$Q = \frac{1}{2\zeta} \tag{19}$$

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Substituting into Equation 10 and solving for Q gives us

$$Q = \frac{1 + \left(\frac{R_3}{R_2}\right)}{3} \tag{20}$$

For design we need the resistor ratio.

$$\left(\frac{R_3}{R_2}\right) = 3Q - 1 \tag{21}$$

Depending on which is more convenient we might choose R_3 and calculate the required value of R_2 or choose R_2 and calculate the required value of R_3. The resistors are often determined so that the sum between about 5,000 and 20,000 ohms.

The range of Q that we can design for is 0.333 for the case where R_3 is zero up to theoretically infinity if R_3 is open but the practical limit is in the range of several tens. High Q systems are very sensitive to small errors or drifts in component values. For simple second-order low-pass or high-pass filters we often use a Q of 0.707 as that results in the flattest response in the pass-band.

For band-pass and band-reject filters we know the natural frequency (same as center frequency) and the desired bandwidth, BW, in Hertz. At the bandwidth edges the magnitude of the transfer function has fallen by 3 dB compared to the peak response at the center for a band-pass filter or the non-reject region for a band-reject filter. The required Q is calculated as

$$Q = \frac{F_n}{BW} \tag{22}$$

See the following link for more information:

http://en.wikipedia.org/wiki/Electronic_filter_topology