Chebyshev Low-pass Filter Math

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March 6, 2013

Cheybshev filters have ripples in the pass-band. If the order is even then the ripples are above the normalized unity DC response. If the order is odd then the ripples are below the normalized unity DC response. Examples are shown in Figures 1 and 2.

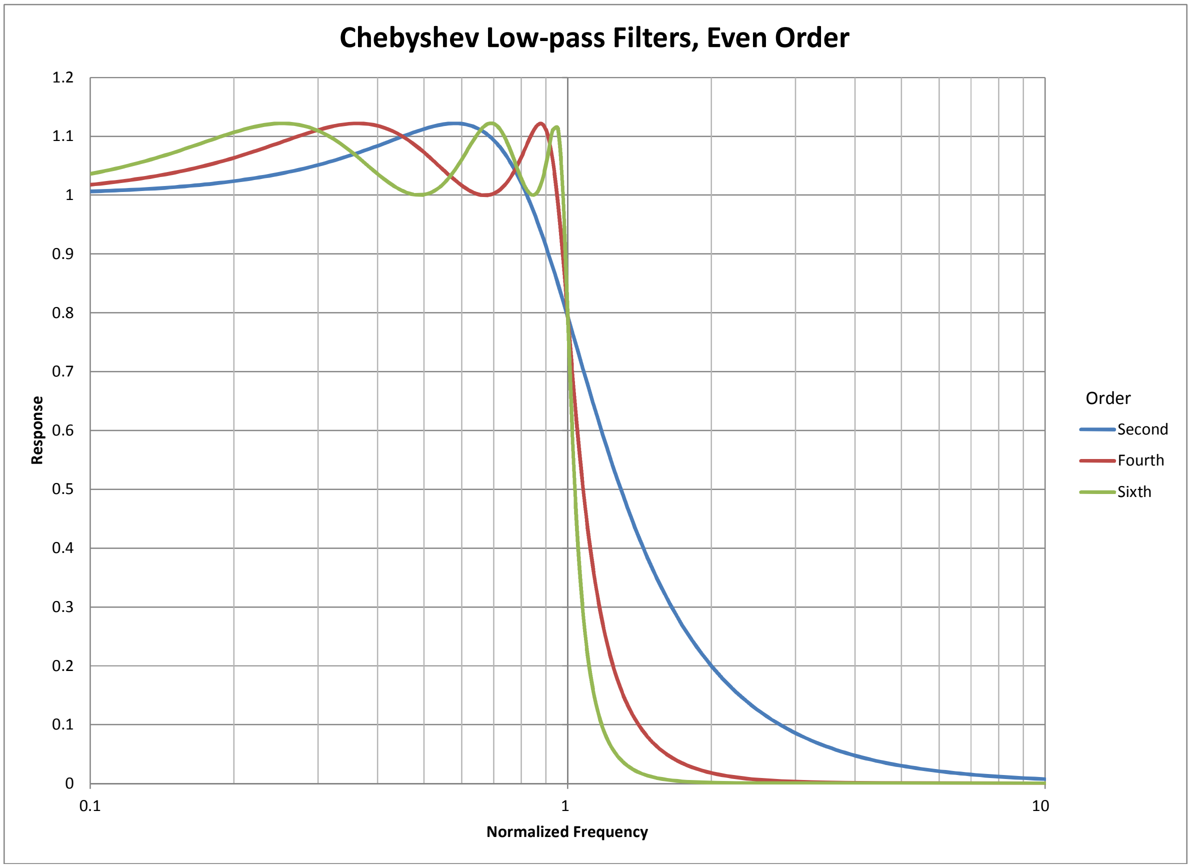

Figure 1: Examples of 1 dB Chebyshev Even Order Low-pass Filter Response

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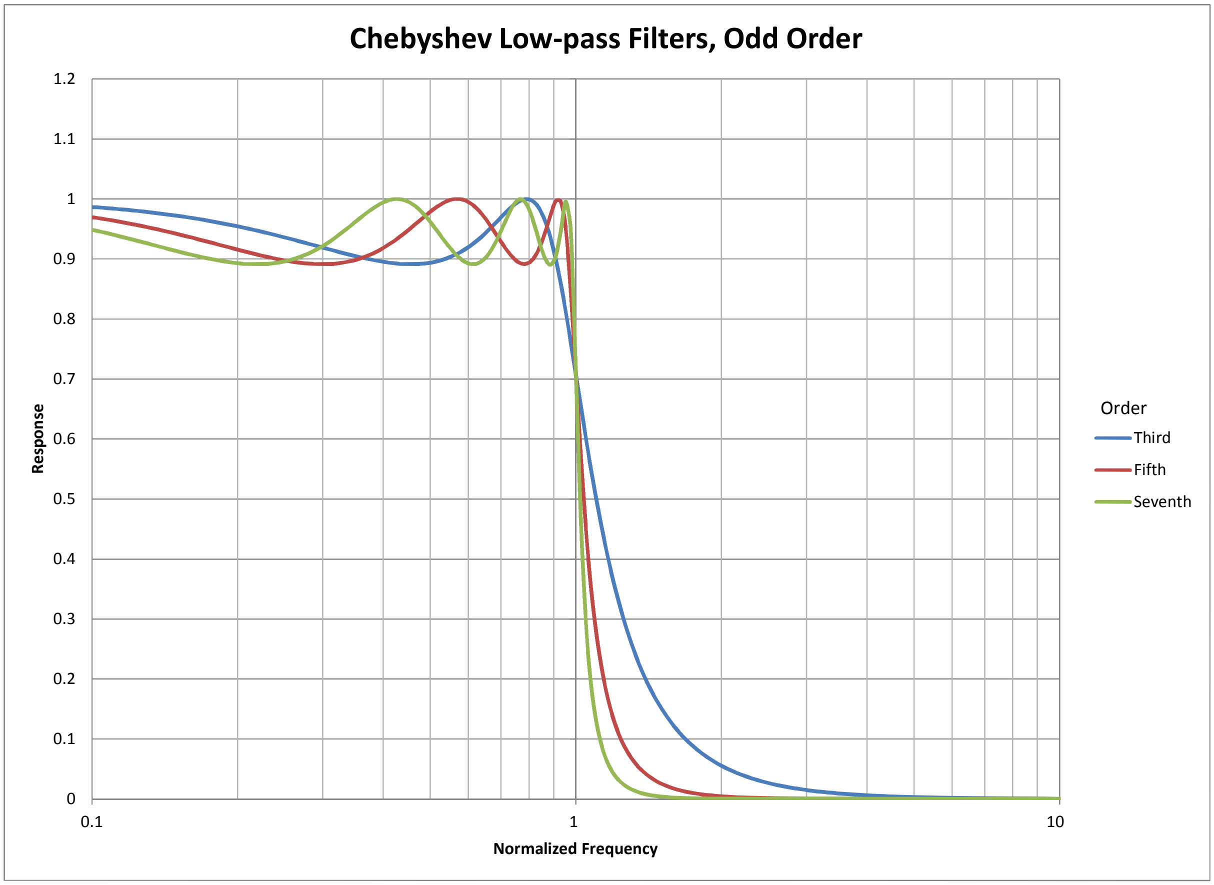

Figure 2: Examples of 1 dB Chebyshev Odd Order Low-pass Filter Response

In determining the required filter order to implement a set of specifications we do not initially know whether the order will be even or odd. In the scheme of things it makes no difference. Therefore, we always specify R_P to be less than unity – typically in the range of around 0.89 to 0.99. The peak-peak magnitude of the ripple will be the difference between 1 and R_P. Our process is to assume an odd order and apply Figure 3. All works fine if the order turns out to be even as what we are most interested in is the uniformity of the response in the pass-band rather than absolute response. Even or odd will have identical uniformity.

To determine the minimum order of a Chebyshev low-pass filter the first step is to specify F_P, R_P, F_S, and R_S as shown in Figure 3. The -3dB cutoff frequency is shown for reference but that is not normally used in the calculation. For ease of illustration F_C is shown a bit higher in relation to F_P than is typical – see Figures 1 and 2.

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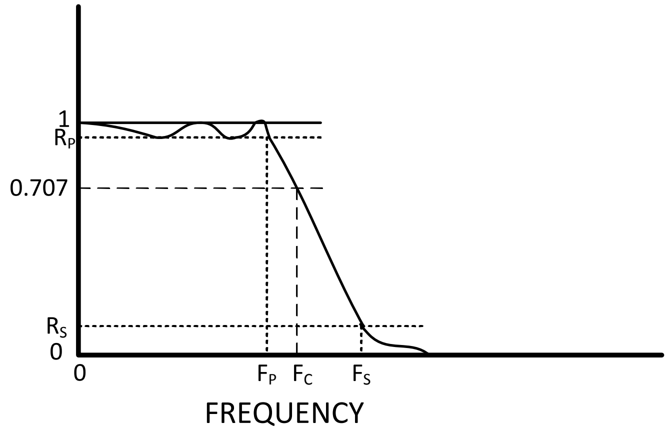

Figure 3: Specifying points for design

First we calculate epsilon using R_P as shown in Equation 1 as we will need that for calculating *n* and also in calculating the pole locations.

$$\epsilon = \sqrt{\frac{1}{R_P^2} - 1} \tag{1}$$

Now we can calculate the minimum order, *n*, that will meet our specifications.

$$n = \frac{\cosh^{-1}\left[\left(\sqrt{\frac{1}{R_S^2} - 1}\right)\Big/\epsilon\right]}{\cosh^{-1}(F_S/F_P)} \tag{2}$$

Generally we round *n* up to the next higher integer particularly if it is close as in rounding 5.83 to 6. If the specifications can be relaxed a bit then rounding down is preferred to limit complexity particularly for cases as in rounding 5.12 down to 5. Keep in mind that the specifications generally contain a certain amount of arbitrary thoughts which typically are more strict than really necessary. You do not want to build a more complex filter than required. It often takes some iteration on the specifications to find the best compromise.

Once we know *n* the poles of the filter are found as follows. The real and imaginary parts of equation 4 are shown on two lines because of the complexity of the equation. Note that the real term is always negative (as it should be – poles must be in left-hand s-plane) as the cosine term is of an angle greater than π/2. All of the poles will be quadratic except that there will be a single pole on the real axis if *n* is odd.

Note that $\omega_P = 2\pi F_P$ (3a)

If we want to calculate directly for the -3dB frequency we can use $\omega_P = 2\pi F_C$ (3b)

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We calculate the system poles using Equation 4 by letting k go from 0 to n-1 although by symmetry we only have to compute to n/2 (and rounding any 0.5 result up to the next integer).

$$pole(k) = \omega_P \cos\left[\frac{\pi}{2}\left(1+\frac{2k+1}{n}\right)\right]\sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right)$$
$$+j\omega_P \sin\left[\frac{\pi}{2}\left(1+\frac{2k+1}{n}\right)\right]\cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right) \qquad (4)$$

The pole locations for a 4^{th} order 1 dB Chebyshev low-pass filter are shown in Figure 4. The numbers by the poles refer to the above equation with k = 0, 1, 2, 3 – actually only the solutions for k = 0 and 1 need to be determined since poles 2 and 3 are symmetric. Note that the poles lie on an ellipse whose major axis is twice the -3dB frequency, ω_C. The ellipse would be slightly taller if ω_P was used. The minor axis dimension depends on ϵ.

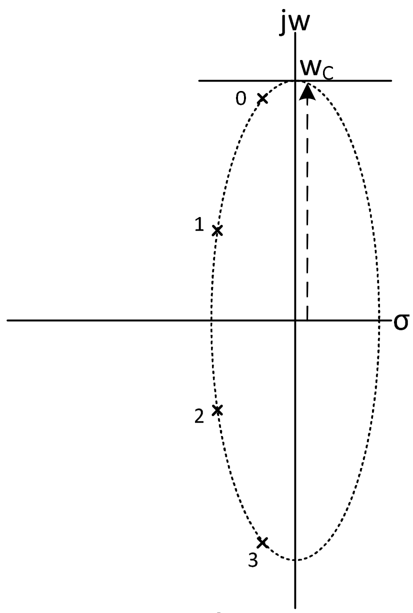

Figure 4: Pole locations of 1 dB 4^{th} order Chebyshev low-pass filter

By definition the cutoff frequency of a Chebyshev filter is when the response falls below R_P. The true -3 dB frequency is a bit mathematically involved to calculate but is only slightly higher than the Chebyshev definition because of the very steep drop in response above the cutoff frequency particularly if the order is four or greater – see Figures 1 and 2. Note that the coefficients in some filter tables have been adjusted so that the -3dB frequency is the given cutoff frequency rather than the frequency at which the response falls outside of the ripple band. When in doubt always use a computer to simulate the response prior to any construction.

See the following web link for more information.

http://en.wikipedia.org/wiki/Chebyshev_filter