Thermal Voltage Noise

by Kenneth A. Kuhn
Feb. 17, 2013

The effect of temperature on a mass is to cause its atoms to vibrate with an intensity proportional to its temperature. This is referred to as thermal agitation. For any conductor at a temperature above absolute zero the electrons loosely held to the nucleus are occasionally shaken loose by thermal agitation thus leaving behind a positive ion. The free electrons wander randomly about and eventually join a positive ion. At any given instant of time the distribution of the electrons is not perfectly uniform across the material and there is a voltage gradient that randomly varies in amplitude and polarity in proportion to thermal agitation. The average value of this voltage is zero. We refer to this varying random voltage as thermal noise or Johnson noise.

Figure 1 illustrates the case of a conductor at absolute zero temperature. With no thermal agitation the distribution of charges is uniform and stationary. There is no voltage across the material.

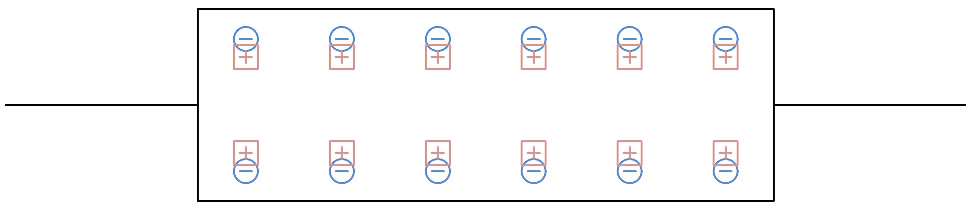

Figure 1: Perfect charge uniformity at zero kelvin results in zero noise voltage

At temperatures above absolute zero the electrons shaken loose can wander about and at any instant of time the distribution may be either more towards one end of the conductor or the other as illustrated in Figures 2 and 3. The location of the positive ions remains fixed. In all cases the net charge on the material is zero.

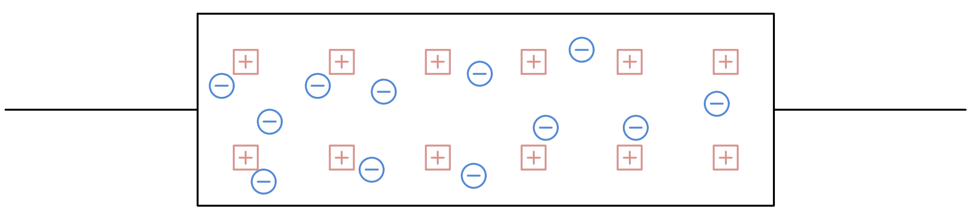

Figure 2: Non-uniform charge distribution results in voltage gradient

Thermal Voltage Noise

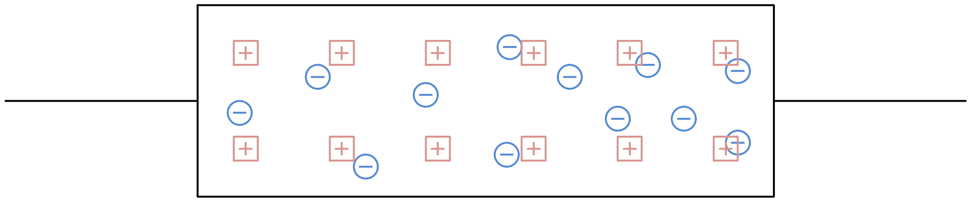

Figure 3: Non-uniform charge distribution results in voltage gradient

The rms voltage across this material is given by Equation 1. The derivation of this equation is beyond the scope of this course.

$$V_n = \sqrt{4kTBR} \tag{1}$$

where:

- V_n is the open-circuit AC rms noise voltage across the material
- k is Boltzman's constant, 1.38E-23 joules/kelvin
- T is the temperature in kelvins
- B is the noise bandwidth in Hertz of the measurement
- R is the resistance of the material in ohms

As an example, the open circuit voltage across a 1,000 ohm resistor at 300 kelvins (room temperature) over a noise bandwidth of 10 kHz is 407 nV.

It is instructive to perform a unit analysis on Equation 1. We note that k has units of joules per kelvin, T has units of kelvins, and B has units of the reciprocal of time or 1 / seconds. When those terms are multiplied the kelvin units cancel and we are left with joules per second which we more commonly refer to as power in watts. Thus the product, kTB, is known as thermal power – that is an important point. The noise bandwidth is always somewhat wider than the electrical bandwidth at the -3dB points to account for the total noise contribution above the cutoff frequency. From a combination of Ohm's and Watt's laws, we know that $V = \sqrt{PR}$. It should now be obvious that Equation 1 has units of volts.

Thermal Voltage Noise

The thermal noise voltage across the resistor is random as shown in Figure 4.

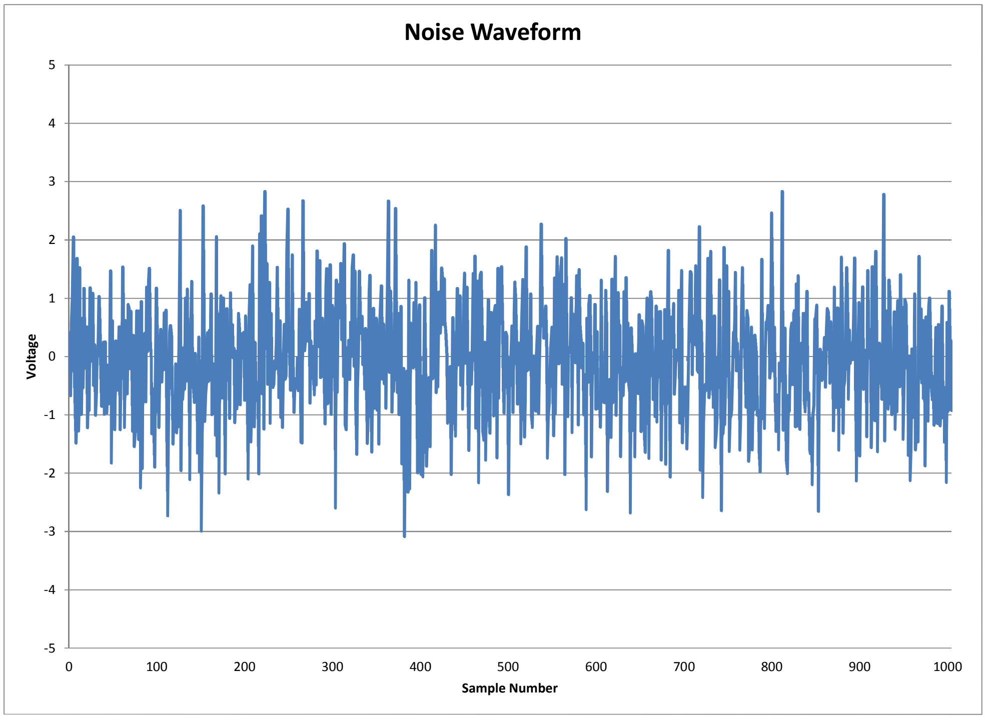

Figure 4: Waveform of random noise

Figure 5 shows a histogram analysis of the distributions of the various instantaneous voltages of the waveform in Figure 4.

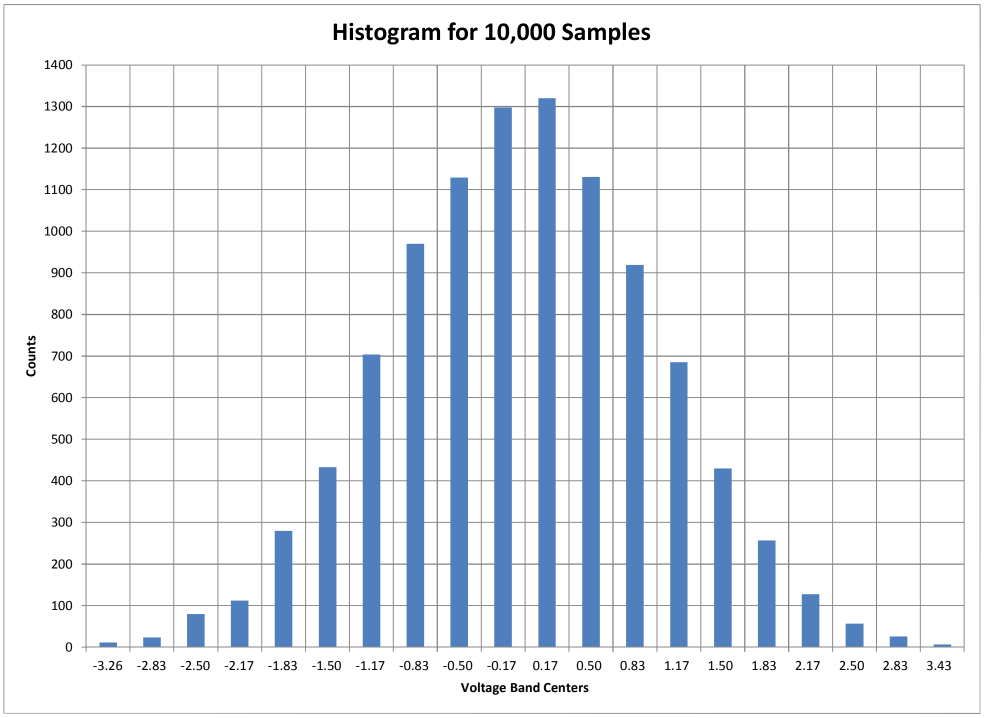

Figure 5: Histogram of random noise

Thermal Voltage Noise

The individual spokes of Figure 5 randomly vary a bit as the process is re-run and over a long time converge to a Gaussian distribution as illustrated in Figure 6. Figure 6 shows the distribution of random voltages for a 1.00 Vrms noise voltage. Observe that the standard deviation of that plot is 1 – the same as the rms noise voltage. Observe that the vast majority of voltages are within +-3 standard deviations although some voltages could extend to numerous standard deviations albeit with a very low probability. The fact that the vast majority of voltages are within +-3 standard deviations leads us to a simple method of dividing the main body of the peak-peak noise voltage observed on an oscilloscope by six to estimate the rms noise voltage.

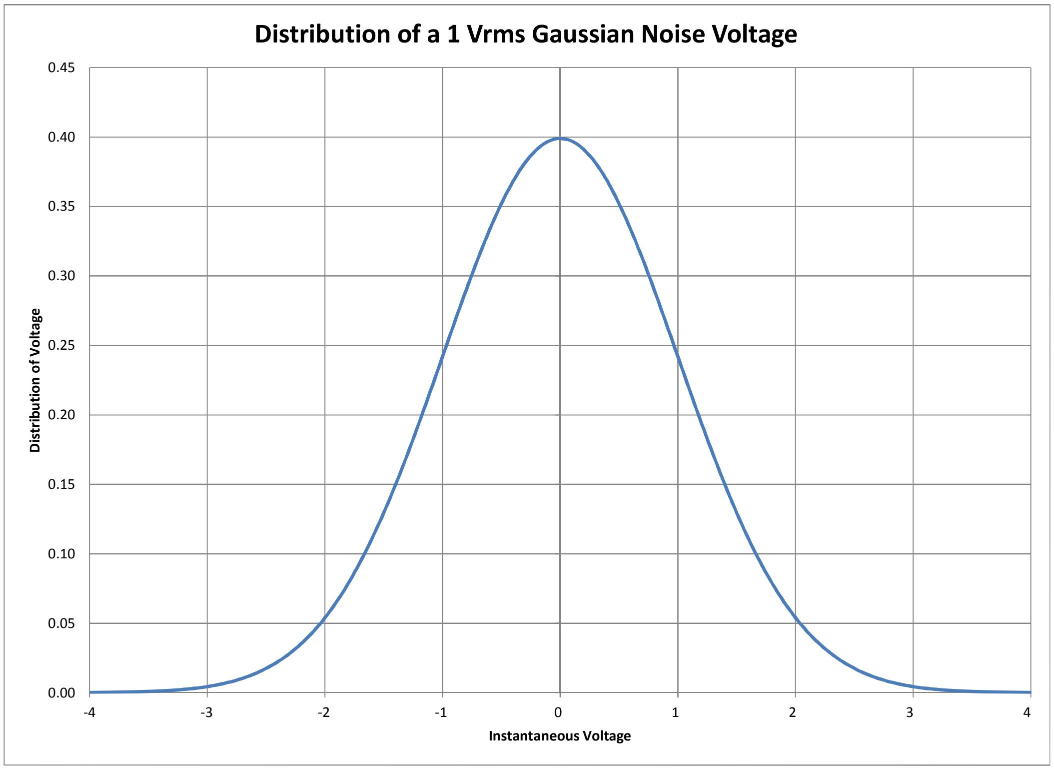

Figure 6: Distribution of voltages of 1 Vrms noise signal

The product kTB is the power that could be delivered to a matched load due to the noise voltage, V_n. No power is dissipated in R_S. Consider Figure 7 where R_S is a resistor with thermal noise, V_n, and R_L is a theoretical noiseless resistor (think of superposition). R_L is equal to R_S for maximum power transfer.

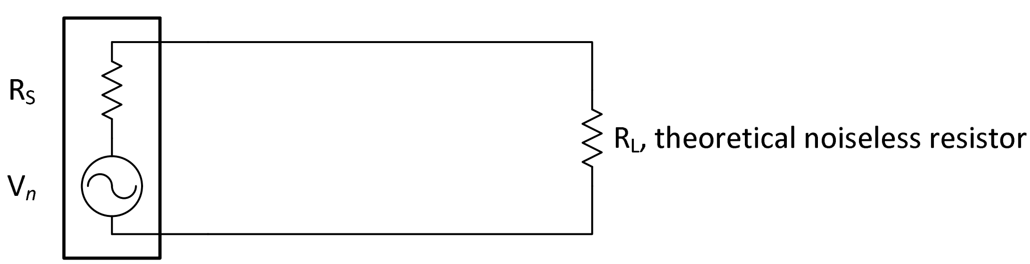

Figure 7: Circuit to aid understanding of thermal power transfer

Thermal Voltage Noise

The load resistor, R_L, receives power, kTB, from R_S. The voltage across R_L is $V_n/2$ because of voltage division. Thus $(V_n/2)^2/R_L = PR_L = kTBR_L$. We let $R = R_L = R_S$. Therefore, $V_n = \sqrt{4kTBR}$. The 4 in Equation 1 comes from this analysis.

In the previous example the power delivered to a 1,000 ohm load resistor would be kTB or 41.4 atto-watts (atto is 10^{-18}). That might seem like a minuscule power that can be ignored but that would be a huge mistake. Thermal noise represents a lower limit to the amplitude of what signals we can process. That is the reason why highly sensitive radio receiving electronics for receiving faint signals from distant space probes are cooled to very low temperatures.

One interesting thing to think about is that if R_S is at a higher temperature than R_L then energy is transferred from R_S to R_L. That means that energy from the thermal environment of R_S is transferred to R_L thus raising its temperature and coupling that energy to the environment of R_L. This is ordinarily a miniscule energy transfer but one should wonder what practical applications might make use of this. Given sufficient time (could be extremely long) and with no external energy sources then the temperatures of the two environments would eventually equalize.

Noise voltage spectral density

An alternate form of Equation 1 is often used for cases where the noise bandwidth is either not yet known or may be varied and we want an easy way to compute the resulting noise voltage. We simply divide Vn by the square-root of the noise bandwidth. This result is known as the noise voltage spectral density and has units of $Volts/\sqrt{Hz}$ and spoken as volts per root Hertz.

Let us use the earlier example of a 1K resistor at 300K over a 10 kHz noise bandwidth and obtaining a total noise voltage of 407 nVrms. Dividing that by the square-root of 10 kHz we obtain a noise voltage spectral density of 40.7 nV/$\sqrt{Hz}$. A new question is what would be the noise voltage over a 1 MHz noise bandwidth? The solution would be

(40.7 nV/$\sqrt{Hz}$) * $\sqrt{1{,}000{,}000}$ = 40.7 uVrms.

When selling devices that have noise specification (such as amplifiers) the manufactures have no way of knowing what bandwidth the customer will be using. Specifying the noise in terms of spectral density solves that problem and the customer can easily determine what noise voltage to expect.

Noise bandwidth versus electrical bandwidth

The electrical bandwidth of a system is generally defined as that frequency where the magnitude response is reduced to 70.7 percent ($1/\sqrt{2}$) of the main response. This is also known as the half-power or -3 dB frequency because power is proportional to voltage squared. The noise bandwidth will always be wider than the electrical bandwidth because noise voltage contributions at frequencies higher than the cutoff will add (albeit a reducing amount) to the total noise voltage.

To determine the noise bandwidth of a low-pass filter we integrate over a frequency range from DC to infinity the square of the noise voltage spectral density multiplied by the frequency response. We take the square root of that result to determine the rms noise voltage. Then we divide the rms noise voltage by the noise voltage spectral density and square the result to obtain the noise bandwidth. This process normalizes so that noise bandwidth is only a function of the electrical bandwidth and not a specific noise spectral density. If we divide the noise bandwidth by the electrical -3dB bandwidth we then have a factor that we can always use henceforth without having to perform the integral. Higher order filters have lower factors because less noise is accumulated because of the steeper slope of the response curve. These factors have been tabulated as follows:

First-order	1.57 (π/2)
Second-order	1.22
Third-order	1.15
Fourth-order	1.13
Fifth-order	1.11

Capacitor "noise"

Capacitors are interesting components which depending on their construction have a number of odd characteristics such as acting as microphones that generate a voltage in response to sound or vibration. Those undesired signals are often referred to as "noise" but are not noise in the sense of this discussion. The following is an interesting link that summarizes some important things to know about capacitors:
http://www.millertechinc.com/pdf_files/TN095%20Capacitor%20Noise.htm .

There is an unexpected and interesting phenomenon known as capacitor thermal noise voltage although capacitors do not produce random noise. Consider the following common circuit consisting of a capacitor in parallel with a resistor. The purpose of the capacitor is to low-pass filter the noise voltage from the resistor in order to reduce the noise.

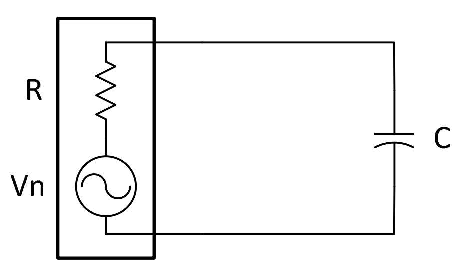

Figure 8: Capacitor in parallel with a resistor

Thermal Voltage Noise

The noise bandwidth of that first-order filter is

$$B = \frac{\pi/2}{2\pi RC} \tag{2}$$

where

B is the noise bandwidth in Hertz
R is the resistance in ohms
C is the capacitance in farads

We know from Equation 1 that the open-circuit voltage across the resistor is $V_n = \sqrt{4kTBR}$. When we substitute Equation 2 into Equation 1 and simplify we have what is known as capacitor noise – although the noise voltage is not actually originating in the capacitor as discussed below.

$$V_n = \sqrt{kT/C} \tag{3}$$

We should perform a units check of what is inside the radical. We note that capacitance has units of coulombs per volt and that voltage has units of joules per coulomb. So we have (joules/kelvin) * (kelvins) / (coulombs/volt) = volt^2. So everything checks out as it should.

What is interesting is that the noise voltage is independent of the resistance. The noise bandwidth and spectral noise density are dependent on the RC product as we expect. It is interesting to ponder what happens at the extremes – where R = zero or R = infinity.

As R is made smaller the noise bandwidth increases and the spectral noise density decreases since Vn remains constant. At R = zero then the noise bandwidth is infinity and the spectral noise density is zero. Thus, over any finite bandwidth the noise voltage is zero.

As R is made larger the noise bandwidth decreases and the spectral noise density increases since V_n remains constant. At R = infinity the noise bandwidth is zero. With zero noise bandwidth the noise voltage is zero.

Thus, the capacitor itself is not producing a thermal noise voltage. What is known as capacitor noise is a consequence of thermal noise from a resistor over a finite bandwidth. Equation 3 tells us that the noise voltage is reduced by using a larger capacitor – which should have been obvious to us all along since that lowers the noise bandwidth. The key point is that Equation 3 tells us that the noise voltage we do end up with over the noise bandwidth is only a function of the capacitance. It does not tell us that the noise originates in the capacitor as a literal interpretation of Equation 3 implies. Be careful about making too literal interpretations. Sometimes some very interesting processes may not be visible as in this case.